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Circle Geometry

Overview

This unit is a breather between the skills-oriented *Geometry Basics* unit and the challenging *Proofs* unit. It is an interesting mix between geometric and algebraic thinking. The unit is essentially broken up into two parts: a group worksheet that introduces the basic circle geometry theorems, and problem sets that practice the implementation of these theorems and their corresponding formulas by solving many problems.

Pacing for this unit

Once the students get the hang of it, the problem sets can be worked through relatively quickly; generally students find it pleasant and satisfying work. The real question is how much time to spend on the group worksheet. Depending on how the teacher approaches it, the group worksheet could take anywhere from two classes (90 minutes) to ten classes.

The group worksheet – discovering the theorems!

Ideally, if time allows, the students are led to discover the theorems for themselves. As mentioned above, it may take several classes to work through this group worksheet, but it is well worth it. It may be best to save the second half of the group worksheet until the students have worked through the first few problem sets.

Beware of the quadratic formula!

There are a few problems where the student is required to solve for an outside segment. This leads to a quadratic equation, and in some cases requires the use of the quadratic formula.

Lesson plan outline for the entire unit

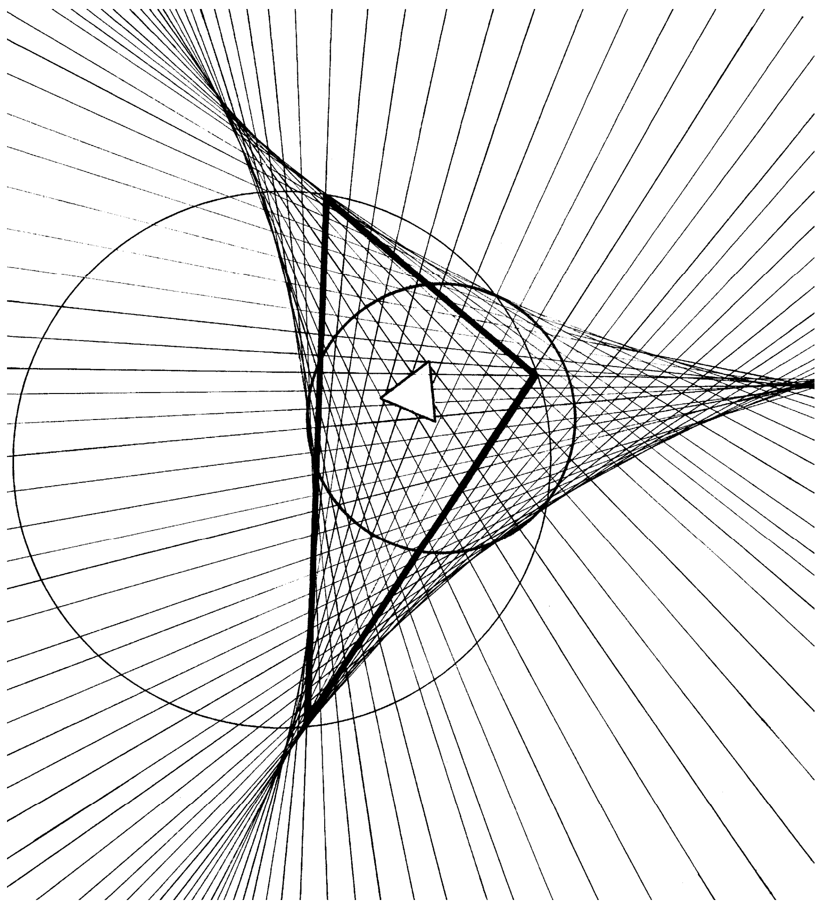
Here is a possible outline of the steps to make this unit successful:

- <u>Definition of terms</u>. This includes *arc*, *chord*, *secant, tangent, inscribed angle, central angle, subtended*.
- <u>First theorems</u>. These are the *Parallel Chord Theorem* and the *Equal Chord Theorem*. These are so simple that the students can clearly see that they are true. We will use these theorems to prove other more complicated ones.
- <u>Discovering the *Intersecting Chord Theorem*</u>. The group worksheet guides the students toward making this discovery through measuring angles and making a conjecture about what the formula might be. This could take two full classes to accomplish – time well spent!
- <u>Proving the *Intersecting Chord Theorem*</u>. There are multiple ways to prove this theorem, two of which are shown in our *High School Source Book*. We recommend the "movement" proof.

Steiner's hypocycloid

This serves as the grand finale of the whole unit. It is a complicated drawing requiring great care, accuracy, and time.

This drawing builds upon the previous drawing. By the end of the drawing – which consists of 60 Simson lines – we come to realize that for any given triangle all of its Simson lines form a hypocycloid. The hypocycloid is tangent to both the nine-point circle and the original triangle, and has the reverse orientation from its Morley's Triangle. Astounding!



Geometry Basics

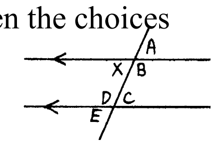
Teachers: Read the commentary on this unit in the introduction.

Problem Set #1

Group Work

Note: In this unit you need a compass, protractor, and ruler.

Look at the drawing below. With #1 through #5, fill in the sentence with the proper words given the choices below.



- vertical angles
- supplementary angles
- corresponding angles
- alternate interior angles
- same-side interior angles

1) X and B are ________.
2) X and C are ________.
3) X and A are ________.
4) X and D are ________.
5) X and E are ________.

Angle Theorems

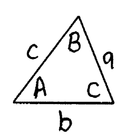
Complete the sentence either with "are congruent" or "add to 180°."

6) *Vertical Angle Theorem.* Vertical angles always…
7) *Supplementary Angle Theorem.* Supplementary angles always…
8) *Corresponding Angle Theorem.* Corresponding angles always…
9) *Alternate Interior Angle Theorem.* Alternate interior angles always…
10) *Same-Side Interior Angle Theorem.* Same-side interior angles always…
11) *Triangle Interior Angle Theorem.* The angles in a triangle always…

Triangle Constructions (by measuring)

With the below table, each row represents a triangle, with three given measurements. Use a ruler and protractor (and perhaps a compass) to construct each triangle on a clean sheet of paper. Then measure (accurately!) to fill in the rest of the table.

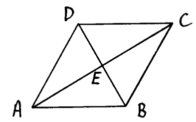
	a	b	c	A	B	C
12)	5cm	7cm	6cm			
13)	7cm	7cm	3cm			
14)	6cm	4cm				110°
15)			7cm	40°	20°	
16)			7cm	40°		120°
17)	7cm			40°	20°	



15) Rhombus Diagonal Theorem.

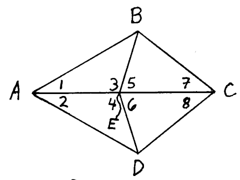
a) What property can be stated regarding the diagonals of any rhombus? Prove it.

b) Given rhombus ABCD, and AB = BD = 8, find the area of the rhombus, and the area of the rectangle formed by BD and AC?

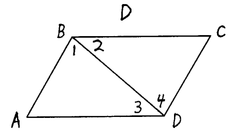


c) For any rhombus, what is the ratio of the area of the rhombus to the area of the rectangle formed by the diagonals?

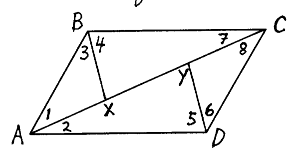
16) Given: AB ≅ AD; BC ≅ CD.
Prove: BE ≅ ED.



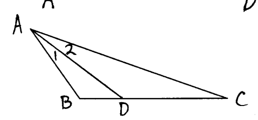
17) Given: AB || CD; AB ≅ CD.
Prove: BC || AD.



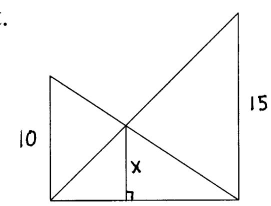
18) Given: ▱ABCD; ∠3 ≅ ∠6.
Prove: ΔADY ≅ ΔCBX.



19) **Triangle Angle–Bisector Theorem.
(from *The Elements,* Theorem VI-3)
"An angle-bisector of a triangle divides the opposite side into segments proportional to the other two sides." Prove it.



20) ** With the diagram shown here, the lines marked as 10, 15, and x are all parallel. It is interesting to note that if the base of the figure were lengthened or shortened, the value of x would not change. Find x.



21) **Eratosthenes' Measurement of the Earth

Around 230 B.C., Eratosthenes calculated the circumference of the earth by using only two measurements. First, at summer solstice, when the sun could shine to the bottom of a well in Syene (which is on the Tropic of Cancer), he measured that the sun at Alexandria (nearly directly north of Syene) was 7.2° off from being directly overhead. Secondly, he calculated, based upon the time it took a camel train to go from Alexandria to Syene, that the distance between the two cities was 5000 stadia.

a) What was Eratosthenes' value for the earth's circumference?

b) Given that the actual (average) radius of the earth is about 6371km, and that 1 stadium is believed to be about 157m (and this is disputed!), what was Eratosthenes percent error?

Problem Set #7

Simplify.

1) $(2x^3y^2)^2(-2xy^3)^3$

2) $(x + 2)(x - 2)^2$

3) $\frac{x^2 - 10x + 21}{x^2 + 10x - 39}$

4) $\frac{3}{4x^2y} + \frac{5}{6xy^3}$

5) $\frac{\frac{\frac{25}{x} - x}{x} - 3}{2x^2 - 5x}$

Factor.

6) $x^2 + 25x - 70$

7) $5x^2 + 25x - 70$

8) $x^2 - 100$

9) $x^6 - 10$

10) $x^4 + 25$

11) $9x^4 - 25x^6$

12) $5x^2 + 23x - 10$

Solve for X in terms of Y.

13) $Y = \frac{2}{3}X - 6$

Find the Common Solution.

14) $x + 2y = 1$
$3x - 4y = 23$

Solve.

15) $\frac{x-2}{2x-25} = \frac{3}{x+20}$

16) $5x + 2 = -3x^2$

17) $(x - 3)^2 = 3x^2 + 4x + 12$

18) $\frac{4x}{x-2} = \frac{x-5}{x-3}$

19) $-\frac{3}{5}X - 4 = \frac{5}{6} - X$

20) $(2x+3)(2x-3) = -x - 6$

21) $\frac{x}{x+1} + \frac{1}{x-2} = \frac{3}{x^2-x-2}$

22) $\frac{2}{x-1} + \frac{8}{1-x^2} = \frac{3}{x+1}$

23) $10x = 3x^2 + 8$

24) $100x^2 + 5x^4 = 6x^4$

25) Find the area of the isosceles right triangle triangle that has a perimeter of 10. (leave answer in radical form.)

26) Bill jogged around a track averaging 3 m/s on the first lap and then 4 m/s on the second lap. The combined time for the two laps was 3 minutes 30 seconds. What is the perimeter of the track?

Problem Set #2

Intervals between Notes

Early in the quest for the hidden mathematical laws behind the beauty of music, the Pythagoreans discovered a very important property – that the most pleasing musical intervals were those that had frequencies that could be reduced to simple whole number ratios. The simpler the ratio (e.g., 3:2), the more consonant, or harmonious, the interval. Intervals with more complex ratios (e.g., 256:243) would sound more dissonant to one's ear.

We can expand further what the Pythagoreans started by creating a complete musical scale based only on simple ratios. The table below shows the basic intervals.

Table of Intervals

Name of Interval	Ratio of Interval	Example
Second	9:8	C → D
Major 3rd	5:4	C → E
Fourth	4:3	D → G
Fifth	3:2	A → E
Major 6th	5:3	C → A
Major 7th	15:8	C → B
Octave	2:1	B → B'

For each resulting note, give the name and frequency (number of vibrations per second). Use the *Table of Intervals* as needed.

1) What is the frequency of the note that is a fifth up from A 440 Hz?
2) What is the frequency of the note that is a fourth up from A 440 Hz?
3) What is the frequency of the note that is a third up from F 352 Hz?
4) What is the frequency of the note that is an octave up from F 352 Hz?
5) What is the frequency of the note that is three octaves up from F 352 Hz?
6) What is the frequency of the note that is two octaves below F 352 Hz?
7) What is the frequency of the note that is 2 fifths above A 440 Hz?
8) What is the frequency of the note that is 2 fourths above A 440 Hz?
9) What is the frequency of the note that is a fifth and then a fourth above A 440 Hz?
10) What is the significance of the answer to the previous problem?

Math & Music ANSWERS

Problem Set #1

The Fifth has a ratio of **3:2**
The Fourth has a ratio of **4:3**
The Third has a ratio of **5:4**
The Sixth has a ratio of **5:3**
The Octave has a ratio of **2:1**

The ratios of these intervals are the same regardless of what note you start on.

Problem Set #2

1) E 660 Hz
2) D $586\frac{2}{3}$ Hz
3) A 440 Hz
4) F 704 Hz
5) F 2816 Hz
6) F 88 Hz
7) B 990 Hz
8) G $782\frac{2}{9}$ Hz
9) A 880 Hz
10) A fifth and a fourth above is an octave.

Problem Set #3

1)

#	Name	Length	Frequency
1.	C	60 cm	264 Hz
2.	D	$53\frac{1}{3}$ cm	297 Hz
3.	E	48 cm	330 Hz
4.	F	45 cm	352 Hz
5.	G	40 cm	396 Hz
6.	A	36 cm	440 Hz
7.	B	32 cm	495 Hz
8.	C	30 cm	528 Hz
9.	D	$26\frac{2}{3}$ cm	594 Hz
10.	E	24 cm	660 Hz
11.	F	$22\frac{1}{2}$ cm	704 Hz
12.	G	20 cm	792 Hz
13.	A	18 cm	880 Hz
14.	B	16 cm	990 Hz
15.	C	15 cm	1056 Hz
16.	E	12 cm	1320 Hz
17.	G	10 cm	1584 Hz
18.	B	8 cm	1980 Hz
19.	C	$7\frac{1}{2}$ cm	2112 Hz
20.	D	$6\frac{2}{3}$ cm	2376 Hz
21.	E	6 cm	2640 Hz

2)

#	Ratio
#2:#1	9:8
#3:#2	10:9
#4:#3	16:15
#5:#4	9:8
#6:#5	10:9
#7:#6	9:8
#8:#7	16:15
#9:#8	9:8
#10:#9	10:9
#11:#10	16:15
#12:#11	9:8
#13:#12	10:9
#14:#13	9:8
#15:#14	16:15
#16:#15	5:4
#17:#16	6:5
#18:#17	5:4
#19:#18	16:15
#20:#19	9:8
#21:#20	10:9

Some of the whole steps are 9:8, and some are 10:9. The half steps are 16:15. But two half steps produce 256:225, which is greater than either 9:8 or 10:9.