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with mistakes. For many students, math teaches them how to (work through) struggle.

- *Combating cynicism*. In today's world, our youth can easily come to believe that things are meaningless and without purpose. Cynicism is a pervasive social disease today. By teaching math in the right way, we can show our students that the world is true and is filled with awe and wonder.
- *Spiritual and moral development*. Rudolf Steiner, the founder of Waldorf education, talks about the importance of math as part of the students' future spiritual development [1]:
  > "The student of mathematics must get rid of all arbitrary thinking and follow purely the demands of thought. In thinking in this way, the laws of the spiritual world flow into him. This regulated thinking leads to the most spiritual truths."
- *Math is human.* Not everything we teach has to be practical. Math, like music, drama, painting, and literature, is an art. Through these subjects, a true and complete education teaches us what it means to be human.

—— *Thoughts on Teaching Math* ——

Phenomenological Math and the Path of Discovery

Often, when a new topic is introduced, math textbooks start off with the finished product – a statement of a theorem, a formula, or a procedure for solving a new type of problem. Following this, examples of the new concept are shown, the students are expected to practice several of these new types of problems, and finally the students are given a test on the topic. This approach essentially feeds the concepts to the students, and leaves them with the impression that math is something that is magically pulled out of a hat. This is the essence of the "blind procedures" approach to math that is so prevalent today (see *Blind Procedures*, above).

I have intentionally reversed the process in this book, and in our workbooks. With a *phenomenological* approach, we begin with a collection of examples, and through the students' own observations, they discern a hidden pattern or shared quality. With the *path of discovery* approach, we serve as the students' guide – being careful not to give too much away. With both of these approaches, students formulate the statement of a rule, property, or theorem for themselves. They create the math (to some degree) rather than having it fed to them.

This book and our workbooks can only be so useful in this regard; ultimately it is part of the art of teaching math to ensure that the students experience math in this real and engaging way.

Balancing skills and mathematical experiences

It is a common misconception that everything we teach should be learned, tested, and retained. Perhaps people think this way because today's math classrooms are dominated by skill development. While I acknowledge that math skills are important, I feel strongly that it is equally important for the students to experience "real math", and to develop mathematical thinking.

It can be helpful to consider that math topics can be divided into two categories: *skills* (i.e., material that needs to be mastered) and *mathematical experiences*.

Skills (a topic that needs to be mastered). Here the teacher needs to create a dance between introducing, deepening, practicing, sleeping, and reviewing. The bigger the topic, the greater the number of times it needs to be put to sleep, and then later reviewed. It is quite typical to introduce a skills topic one year, but not to have the students reach mastery until the next year, or the year after.

Mathematical experiences. While our workbooks allow the students adequate time to practice essential skills, they also create opportunities for discovery. This discovery process is part of a broader category of *mathematical experiences* – something that is critical for developing true mathematical thinking capacities, yet so neglected in today's math classrooms. Mathematical experiences include discovery, puzzles, games, and problem solving. We teach these topics because they stretch our students' minds, they teach mathematical thinking, and they engender enthusiasm and wonder for math. This book and our workbooks can only do so much. *It is ultimately the responsibility of the teacher to create opportunities for mathematical experiences*. As a guideline, I recommend that between 25% and 50% of classroom time be spent on mathematical experiences.

[1] Rudolf Steiner, *Spiritual Ground of Education,* lecture given at Oxford, August 21 1922, GA 305

—— ***Possibility and Probability*** (9th Grade Main Lesson) ——

Overview

The subject of *possibility and probability* (which includes permutations and combinations) concentrates on answering questions such as: How many possible (shortest) routes are there for going between two points marked on a grid? How many ways are there for twenty students to get in line? How many different committees of three can be formed from a group of ten people? What is the probability of flipping five coins and getting all heads?

These questions often yield surprising results. It is through careful thinking that we can overcome the task of making sense of these difficult problems. We recognize patterns and similarities with previously encountered problems and learn to solve the problems in a systematic way.

The students work both independently and in groups to solve problems. Carefully worded explanations of a few solutions are written in their lesson books. Each student is challenged at some point during the course, and hopefully develops confidence in their thinking.

Lesson plans

Detailed day-by-day lesson plans can be downloaded from our website: **www.JamieYorkPress.com**.

The big picture

This block is only intended to be an introduction to this subject area. These problems can get very difficult. In this block we are working with the big picture. We only do a few central problems very thoroughly. The purpose here is not to work on skills; we don't expect the students to become proficient at these problems during this main lesson block. Later in the year, they will revisit this topic in the track class as they work with the *Possibility & Probability* unit in our *9th Grade Workbook*. This is when the students get to practice solving many of these problems. Then in eleventh grade they revisit the topic once again and work with more challenging problems in the *Possibility & Probability* unit of our *11th/12th Grade Workbook*.

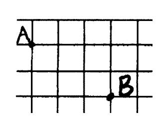
Learning to think independently

At the start of this block, I give a motivational speech about the importance of learning to think for yourself. I make the point that most people today simply believe what they are told (e.g., in the TV news, or on the Internet). I say to the students: "In the lower grades, you were expected to believe what your teacher told you. Your teacher probably told you that π was approximately 3.14. You believed your teacher. It is now time for you to start thinking for yourself." Here is the key thought for our block:

> *"Don't just believe what you're told. Believe it only when you know – in your own thinking – that it's true."*

Central questions

- With each of the below questions, I try to get the students to discover the answer as much as possible on their own; I resist the temptation to give them answers. Students present their ideas to the class, and slowly everyone comes to clarity about the correct solution.
- After they have reached clarity about a solution, they write up the problem in their main lesson books. For each essay, I expect a statement of the question, a description of what they did (even if they made an error), and a full explanation of the correct solution.
- *The Street Problem*. How many shortest routes are there from A to B?



  - I begin the block with this question. The students inevitably draw out all of the possible routes, and usually come up with the correct answer (10).
  - Then I ask how many possibilities would there be for a 5x6 grid? Finding a method to answer this question becomes a goal for the block.
- *The Wardrobe Problem*. How many possible outfits can you choose to wear if you have 3 pants and 5 shirts to choose from?
  - This leads to the *Fundamental Counting Principle* (see the Summary Sheet below).
- *The Seating Chart Problem*. How many possibilities are there for making a seating chart for the class?
  - This leads to the idea of factorial.

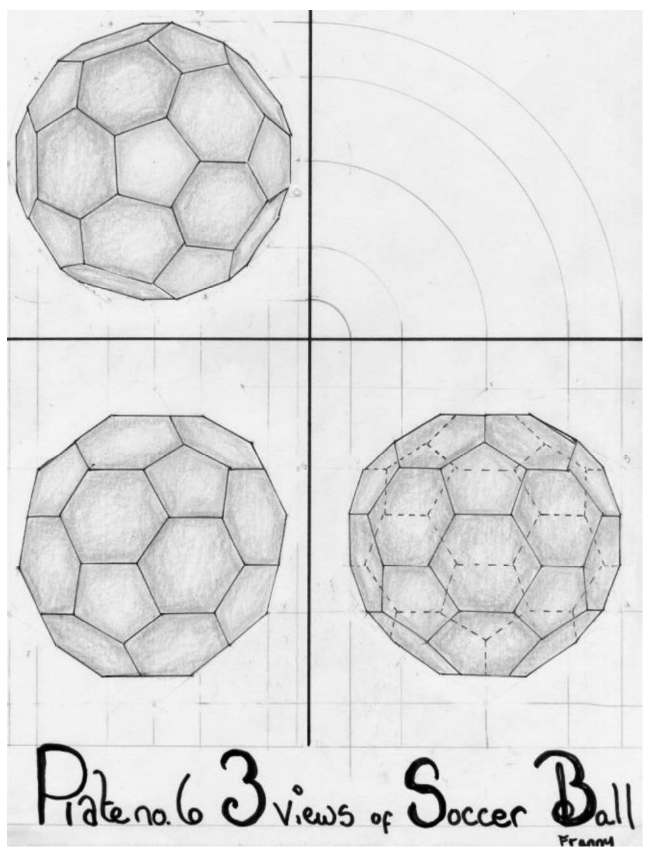
Instructions for Drawing a **Truncated Icosahedron** (a.k.a. Soccer Ball)

Notes for the teacher:

- The truncated icosahedron is quite challenging, and should only be attempted by advanced students.
- It is helpful to make a model of a truncated icosahedron from Zometools (all blue sticks) in order to best visualize what the various views look like.
- The truncated icosahedron has 32 faces, 60 points, and 90 edges. However, once the drawing of the icosahedron has been mastered, the truncated icosahedron is quite workable.
- Conceptually, the truncated icosahedron is the product of "chopping off" the 12 points of the icosahedron, whereby each point is replaced with a regular pentagon and each triangular face of the icosahedron transforms into a regular hexagon.
- Once the students have learned how to draw the dodecahedron and icosahedron, it may be best to have the students figure out a process for drawing the truncated icosahedron. They might figure out a different process than what is shown here.

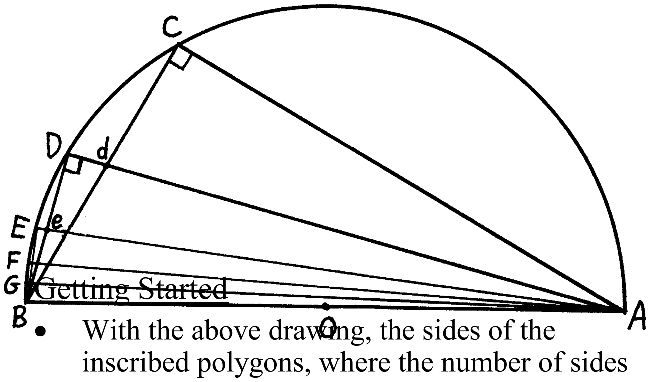
Outline of the process for the drawing:

- Lightly in lead pencil, draw the top view of an icosahedron according to the instructions given for the icosahedron. (Some of the lines will be erased later.)
- With the front view of the icosahedron, the points were located on four different levels shown by horizontal lines. (The top and bottom lines only had one point, so drawing these lines weren't actually necessary. But we will leave them here for reasons that will soon be evident.) Lightly in lead pencil, draw these four lines (with the proper spacing) across the front view of the truncated icosahedron.
- With the front view and the four lines just drawn, draw two more horizontal *evenly-spaced* lines (do you understand why?) between the top two lines (lightly in lead pencil), two more horizontal *evenly-spaced* lines between the middle two lines, and two more horizontal *evenly-spaced* lines between the bottom two lines. Erase the original top and bottom lines. You should now have eight horizontal lines across the front view.
- *Drawing the Top View.* Each of the 30 edges of the icosahedron in the top view (including dotted edges that are in the background) needs to have two evenly spaced points (at distances of ⅓ and ⅔ the length of the edge) marked on it. Carefully and accurately locate these 60 new points with very small dots in black ink. Erase the lead pencil lines in the top view, and then connect the 60 inked points to complete the top view. Start by drawing the small pentagon in the center, and then the five hexagons connected to that pentagon. This top view should have dotted background lines (which aren't shown in the drawing here).
- *Drawing the Front View.* We have already drawn 8 horizontal lines in the front view. Of the 60 points in the front view, the middle four horizontal lines should have ten points each, and the top two and bottom two lines should have five points each. Locate these 60 points and then complete the front view. Notice that there should be no dotted lines in the front view.
- The *Side View* can now be completed.



Archimedes’ Calculation of π – as Archimedes did it

Inscribed Polygons



Getting Started

- With the above drawing, the sides of the inscribed polygons, where the number of sides is 6, 12, 24, 48, 96, are given by BC, BD, BE, BF, BG, respectively.
- AB is a diameter of the circle, and O is the center.
- Our primary goal is to find the ratio of AB:BG. We can then find the ratio of the perimeter of the 96-gon to the diameter of the circle.

Calculating the 6-gon (hexagon)

- Because ΔABC is a 30°-60°-90° triangle, we know that AC:BC = $\sqrt{3}$:1 and AB:BC = 2:1
- Archimedes approximates $\sqrt{3}$ as $\frac{1351}{780}$, which is too large by 0.000027%. We can then say:

AC:BC < 1351:780 and **AB:BC = 1560:780**

Calculating the 12-gon

- With ΔABC, ∠BAC is bisected by AD.
- AC:Cd = AB:Bd [Δ ∠ Bisector Th.; *Elem*. VI-3]
- AC:Cd = AB:Bd = (AC+AB):(Cd+Bd) [*Elements* V-12]
- AC:Cd = AB:Bd = (AC+AB):BC
- AC:Cd = AD:BD [because ΔACd ~ ΔADB]
- AD:BD = (AB+AC):BC [from above steps]
- By using the ratios from the 6-gon, we get: AD:BD < (1560 + 1351) : 780

AD:BD < 2911 : 780

- $AB^2 = AD^2 + BD^2$ [Pythagorean Theorem]
- $AB^2 : BD^2 = (AD^2 + BD^2) : BD^2$
  Using the above ratio, we get:
  $AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$
  $AB^2 : BD^2 < 9082321 : 608400$

AB:BD < 3013¾ : 780

Calculating the 24-gon

- We repeat the same process, but leave out the smaller steps this time.
- With ΔABD, ∠BAD is bisected by AE.
- AD:De = AB:Be [Δ ∠ Bisector Th.; *Elem*. VI-3]
- AE:BE= (AB+AD):BD < (3013¾ + 2911) : 780
  AE:BE < 5924¾ : 780, which reduces to:

AE:BE < 1823 : 240

- $AB^2 : BE^2 = (AE^2 + BE^2) : BE^2$
  $AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$
  $AB^2 : BE^2 < 3380929 : 57600$

AB : BE < 1838 $\frac{9}{11}$: 240

Calculating the 48-gon (with even fewer steps!)

- With ΔABE, ∠BAE is bisected by AF.
- AF:BF = (AB+AE):BE < (1838 $\frac{9}{11}$ + 1823) : 240
  AF:BF < 3661 $\frac{9}{11}$: 240, which reduces to:

AF:BF < 1007 : 66

- $AB^2 : BF^2 < (1007^2 + 66^2) : 66^2$
  $AB^2 : BF^2 < 1018405 : 4356$

AB : BF < 1009 $\frac{1}{6}$: 66

Calculating the 96-gon (finally!)

- With ΔABF, ∠BAF is bisected by AG.
- AG:BG = (AB+AF):BF < 2016 $\frac{1}{6}$: 66

AG:BG < 2016 $\frac{1}{6}$: 66

- $AB^2 : BG^2 < (2016\,{}^1/_6{}^2 + 66^2) : 66^2$
  $AB^2 : BG^2 < 4069284 \frac{1}{36} : 4356$

AB : BG < 2017¼ : 66 Our goal!!

Calculating a Lower Bound for π

- Let P = Perimeter of 96-gon, and
  Let D = Diameter of circle.
- From above: BG : AB > 66 : 2017¼
- π:1 > P : D = 96·BG : AB
- π:1 > 96·66 : 2017¼ = 6336 : 2017¼ > 3 $\frac{10}{71}$: 1

$$\pi > 3\frac{10}{71}$$

—— *Projective Geometry* (11th Grade Main Lesson) ——

Overview

In many ways, projective geometry – a subject which is unique to the Waldorf math curriculum – is the climax of Waldorf students' multi-year study of geometry. It gives the students a completely different experience of geometry. The thinking involved is both demanding and creative. For many students projective geometry is exciting and refreshing.

We start with a philosophical debate. Under some circumstances, it appears that two parallel lines meet. For example, artists during the Renaissance noticed that two lines, which are known to be parallel, actually meet in the drawing. Historically, it took a couple hundred more years before people dared to question Euclid's fifth postulate, which essentially states that two parallel lines never meet. We then decide to work – perhaps somewhat skeptically at first – with the assumption that two parallel lines meet at infinity. What happens then? This leads us to investigate many different theorems in projective geometry, including theorems from Pappus, Desargues, Pascal, and Brianchon. The topics get more sophisticated during the second half of the course as we study the principle of duality, line-wise conics, and conclude with an in-depth study of polarity.

Lesson plans

Detailed day-by-day lesson plans can be downloaded from our website: **www.JamieYorkPress.com**.

What this unit is

In order to cover all of projective geometry, this unit would have to be infinitely long. (I had to have one bad joke in this book.) In the space given here, I can only cover some of the basic ideas of projective geometry, and include some suggestions about what can be done in the classroom.

Advice for those new to projective geometry

I was completely new to projective geometry when I first started as a Waldorf teacher. I found it strange, challenging, and wonderful. I find that projective geometry takes years of slow, careful study before it completely sinks in. But there is no need to feel intimidated. As opposed to many topics in mathematics, where there is a clear list of topics that ought to be covered, projective geometry is vast and offers many possibilities. Projective geometry courses taught by two teachers could cover completely different topics and theorems and yet still accomplish the same goal: to have the students experience sense-free imaginative thinking and get a brief taste of this amazing subject.

If you are new to the subject, I would suggest the following progression: (1) Read through this unit and study the drawings, which will give you an impression; (2) study Olive Whicher's book, *Projective Geometry*; (3) study my projective geometry lesson plans (found on our website); (4) read Lawrence Edwards' book, *Projective Geometry*; and (5) read H.S.M. Coxeter's book, *Projective Geometry*.

Above all, the best way to really learn projective geometry is to take the time to do many, many drawings yourself. Ideally, all of this study would take place over a period of many months or years.

More than pretty pictures

The purpose of PG is *not* to just make pretty drawings. It is important for the students to understand deeply what the drawings represent. The process in the students' imagination is more important than the finished drawing. I want to encourage the students to experiment; not every drawing has to be in "perfect" finished form. Although I am not fond of having students display their work (people can't get much of an impression of what PG is by looking at a PG drawing), I often have the students complete one complicated drawing, such as the polarity of a curve, as a "final project" which gets displayed in my classroom for a while.

Grades and tests

I question the value of grades for any course, but especially for projective geometry, I have found grading very difficult and counter-productive. I could have one student quickly completing many drawings and making them all look beautiful, and then another student thinking deeply about a couple of drawings, experimenting, and only completing a few drawings that might look less beautiful. How do you grade this? For these reasons, I don't give grades in my projective geometry main lesson. Also, end-of-block tests often have questionable real value. For a number of reasons, I don't give a test during this block. (See also "Grades and Tests" under *Thoughts on Teaching Math* in the Introduction)

Cantor's Set Theory

Proof that __the rational numbers are countable__ (which means they can be put into a one-to-one correspondence with the counting numbers):

- Make a 2-D array which encompasses all of the rational numbers in the following way: the first row is $^1/_1$, $^2/_1$, $^3/_1$, etc., the second row is $^1/_2$, $^2/_2$, $^3/_2$, etc., the third row is $^1/_3$, $^2/_3$, $^3/_3$, etc.
- Now we can number all of them along diagonals so that #1 is $^1/_1$, and #2 is $^2/_1$, #3 is $^1/_2$, #4 is $^3/_1$, #5 is $^2/_2$, #6 is $^1/_3$, #7 is $^4/_1$ #8 is $^3/_2$, etc.
- Thus, the rational numbers can be put into a one-to-one correspondence with the counting numbers.

Proof that __the algebraic numbers are countable__:

- Every algebraic number is a (complex number) solution to an algebraic equation in this form: $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0x^0 = 0$, where a_i is an integer, and n is a positive integer.
- Our goal is to find a systematic method for putting all of the algebraic numbers into a specific order, thereby matching them in a one-to-one correspondence with the counting numbers.
- Let A(i) be the i^{th} algebraic number.
- Let $K = n + |a_n| + |a_{n-1}| + |a_{n-2}| + \ldots + |a_1| + |a_0|$, where K represents the "index" of the equation.
- Usually, there are several algebraic equations for a given index, K. Some of these equations have multiple solutions. Some of these solutions will be ignored because they are algebraic numbers that were found with an equation from a smaller index. Of all of the new (complex number) solutions to the equations produced by a given index, we will order them (from least to greatest) according to their complex components, and then their real components.
- There are no equations where K = 0 or K = 1 that yield any algebraic numbers as solutions.
- Of the equations where K = 2, only one algebraic number (x = 0) is produced. Therefore A(1) = 0
- Of the equations where K = 3, only two new algebraic numbers (x = −1, 1) are produced. Therefore A(2) = −1, and A(3) = 1
- Of the equations where K = 4, six new algebraic numbers ($x = \pm 2, \pm\frac{1}{2}, \pm i$) are produced. Therefore $A(4) = -i$, $A(5) = -2$, $A(6) = -\frac{1}{2}$, $A(7) = \frac{1}{2}$, $A(8) = 2$, $A(9) = i$.
- Of the equations where K = 5, twenty new algebraic numbers (including $x = \pm\sqrt{2}, \pm\sqrt{2}\,i, \pm\phi$) are produced, thereby giving us the values of A(10) through A(29).
- This can be continued indefinitely for all values of K.
- Therefore all of the algebraic numbers can be put into a one-to-one correspondence with the counting numbers.

Proof that __the real numbers are not countable__:

- Every real number has a specific and unique decimal representation.
- __Assumption__: The real numbers between 0 and 1 are countable.
- It is therefore possible to write down __all__ of the real numbers between 0 and 1 in a systematic order (but not according to size) from a_1, a_2, a_3, … etc., thereby matching them in a one-to-one correspondence with the counting numbers.
- Let d be a 2-D array such that for any number, a_n, the i^{th} digit of a_n is given by $d_{n,i}$.
  This means that the 4^{th} number in our list, a_4, has a decimal expansion of:
  $a_4 = 0\,.\,d_{4,1}\,d_{4,2}\,d_{4,3}\,d_{4,4}\,d_{4,5}\ldots$ (So if $a_4 = 0.463872048\ldots$, then $d_{4,6} = 2$)
- Now we will create a new number, j, between 0 and 1. We assign its first digit to be anything but $d_{1,1}$ and the 2^{nd} digit to be anything but $d_{2,2}$, and the 3^{rd} digit to be anything but $d_{3,3}$, etc.
- j is not equal to a_1 because they have different first digits; j is not equal to a_2 because they have different second digits; j is not equal to a_3 because they have different third digits, etc.
  j is not equal to any number on our list. We have therefore found a real number between 0 and 1 that is not on the list, so the list does not include all of the real numbers.
- Therefore, our assumption is false; the real numbers are *not* countable.
- __Corollary__: There exists at least one irrational number that is not algebraic. (These numbers are called *transcendental numbers*.)
- __Corollary__: The transcendental numbers are *not* countable. (Loosely speaking, this means that there are "far more" *transcendental numbers*, than there are rational or algebraic numbers.)
- __Corollary__: The transcendental numbers and the irrational numbers have the same order, or *cardinal number* (i.e., they can be mapped into a one-to-one correspondence with each other).