

Table of Contents

INTRODUCTION	7		
<i>About this Book</i>	7		
<i>Today's Challenge</i>	8		
<i>Thoughts on Teaching Math</i>	10		
<i>Practical Advice</i>	13		
<i>Will our Students be Prepared?</i>	10		
<i>Tips for Success in Math</i>	11		
MATH CURRICULUM SUMMARY	18		
Sixth Grade	18		
Seventh Grade	19		
Eighth Grade	20		
SIXTH GRADE ARITHMETIC	21		
The year for strengthening skills	21		
The order of topics	21		
<i>The World of Numbers</i>	21		
Mental Math	21		
Math Tricks	21		
New Multiplication Facts	21		
Casting Out Nines	21		
Exponents and Roots	22		
Divisibility Rules	22		
Prime Factorization	23		
LCMs and GCFs	23		
Common denominators for	23		
<i>Division</i>	23		
Division and Fractions	23		
Think of division as a fraction	23		
Making the divisor easier	23		
Long Division	23		
Vocabulary	23		
Normal long division	24		
Don't leave a remainder	24		
How to know if a digit in the answer is too small?	24		
An explanation of why long division works	24		
Division problems made easier by rounding	25		
Short Division	26		
Checking Answers	26		
<i>Fractions</i>	26		
Fractions aren't just pizza!	26		
Fractions are part of the whole	26		
Reducing fractions	27		
Fractions are division	27		
Mixed numbers	27		
Comparing Fractions and Decimals	27		
Compound Fractions	28		
<i>Decimals</i>	28		
Review from Fifth Grade	28		
Fraction to Decimal Conversions	28		
Fraction/decimal conversions to memorize	29		
The trick for 11ths, and 20ths	29		
Decimal to Fraction Conversions	29		
Repeating Decimals	29		
Determining number of digits under the repeat bar	29		
Division Problems with Repeating Decimals	30		
Converting Repeating Decimals to Fractions	30		
9, 99, etc. in the denominator	30		
90, 990, etc. in the denominator	30		
Repeating decimals into fractions	31		
<i>Business Math Main Lesson</i>	31		
A Few Thoughts on this Unit	31		
Keys to Success for Percents	31		
Don't rely on pictures	31		
Percent to Fraction Conversions	32		
Determining a Certain Percent of a Given Number	32		
Looking at it as a division problem	32		
Converting to a fraction and multiplying	32		
		Converting to a decimal and multiplying	32
		Determining a Percentage	32
		Percent Increase and Decrease Problems	33
		Increasing or decreasing a number by a certain percent	33
		Calculating the percentage of increase or decrease	33
		Profit, Commission, and Tax	33
		Interest	33
		Comparing simple interest and compound interest	33
		Calculating simple interest	33
		Discount and Loss	34
		Rate of Pay	34
		The Unitary Method and Unit Cost	34
		Temperature Conversion Formulas	34
		Business Formulas	35
		Rate of pay	35
		Simple interest	35
		Price after Tax	35
		Discount Price	35
		Graphing	35
		Pie charts	35
		Line graphs	35
		A good page for a main lesson book	35
		<i>Other Topics</i>	36
		US Standard Tools	36
		Metric System	36
		Definitions	36
		Developing a sense for metric	36
		Word Problems	37
		General word problems	37
		Rate of speed	37
		A Key Strategy	37
		Ratios	37
		Comparing ratios and fractions	37
		Statistics	38
		Arithmetic mean	38
		Median	38
		Mode	38
		Significant Digits	38
		Currency Exchange Rates	39
		SIXTH GRADE GEOMETRY	40
		<i>The Basics</i>	40
		Basic Geometry Terminology	40
		Angle Measure	40
		Polygon Terminology	40
		Types of triangles	40
		Types of quadrilaterals	40
		Polygons with more than four sides	40
		Circle Terminology	41
		The Three Dimensions	41
		<i>Geometric Drawing</i>	41
		Tips for doing Geometric Drawings	41
		Copying a Line Segment	41
		Copying an Angle	42
		Bisecting a Line Segment	42
		Bisecting an Angle	42
		A Perpendicular Line through a Point on that Line	43
		A Perpend. Line through a Point Not on that Line	43
		Constructing a Parallel Line	43
		Dividing a Line Segment into Equal Parts	44
		Constructing a Triangle, Given One Side	44
		Constructing a Square, Given One Side	44
		Constructing a Hexagon, Inside a Given Circle	45
		Constructing a Square, Inside a Given Circle	45
		Constructing a Triangle, Inside a Given Circle	45
		Constructing an Octagon, Inside a Given Circle	46
		The 12-Division of the Circle	46
		The 24-Division of the Circle	46

<i>Spirals</i>	47	<i>Word Problems</i>	62
The central idea here	47	Measurement Word Problems	62
The Equiangular Spiral	47	Rate Problems	62
Construction	47	Average rate of speed	62
Interesting questions	47	Compound rate problems	62
Geometric Progressions and the Equiangular Spiral	47		
Other Ways to Construct an Equiangular Spiral	47	SEVENTH GRADE ALGEBRA	63
Formed with inscribed regular polygons	47	Basic Goals	63
Joining the quarter-points of the square's sides	48	The Importance of Form	63
The Spiral of Archimedes	48	History of Algebra	63
<i>Advanced Constructions</i>	49	Terminology	63
Rotations of Circles	49	<i>Formulas</i>	64
The Limaçon and the Cardioid	49	Gauss's Formula	64
The Hierarchy of Quadrilaterals	49	Cost of Renting a Car	64
Knots and Interpenetrating polygons	49	Galileo's Law of Falling Bodies	64
The 24-Division with all its Diagonals	49	Euclid's Perfect Number Formula	65
The King's Crown	49	Using Euclid's formula	65
<i>Area</i>	50	<i>Positive and Negative Numbers</i>	65
Area of square, rectangle, and right triangle	50	A Careful Introduction	65
		Combining Positive & Negative Numbers	65
SEVENTH GRADE ARITHMETIC	51	Multiplication and Division Rules	66
The importance of seventh grade	51	<i>Expressions</i>	66
The order of topics	51	Simplifying Expressions	66
<i>The World of Numbers</i>	51	Law of Any Order	66
Math Tricks	51	Combining like terms	66
Divisibility Rules	51	Fractions/decimals as coefficients & constants	66
Roots	52	<i>Equations</i>	66
<i>Measurement</i>	52	An Equation is a Puzzle	66
Review	52	An Introductory Puzzle	66
The Metric System	52	Solving by Guess and Check	67
The metric stairs	52	The Golden Rule of Equations	67
Conversions in a Given System	52	Solving Equations by Balancing	67
<i>Percents</i>	53	<i>Algebraic Word Problems</i>	68
Finding the Base	53		
Easier ones	53	SEVENTH GRADE GEOMETRY	69
Thinking of inverses	53	<i>Area</i>	69
Trickier ones	53	The Shear and Stretch	69
Strange Percents	53	Area of a parallelogram	69
Compound interest	54	Area of a non-right triangle	69
Calculating the Percentage of Increase or Decrease	54	<i>Geometric Drawing</i>	69
<i>Ratios</i>	55	Triangle Constructions	70
Key Ideas	55	SSS	70
Ratios have no units	55	SAS	70
Ratios of more than two things	55	ASA	70
The Three Thoughts of a Ratio	55	SSA	71
The Two Forms for a Ratio	56	AAS	71
Whole number form	56	Euclidean Constructions	72
Decimal form	56	The three rules of the game	72
The Two Thoughts of a Ratio	56	Why compass and straightedge only?	72
Reciprocals of Ratios	56	Various Methods for doing Constructions	72
Proportion of the Whole	57	Constructions with compass and straightedge	72
Similar Figures	57	Measurement constructions	72
Shadow problems	58	The Guess and Check Method	72
Direct and Inverse Proportions	58	Approximate Constructions	72
Speed, time, and distance	58	Geometric Division	73
String length and frequency	58	Geometric division of a 12-gon	73
The law of the lever	58	Geometric division of a 15-gon	73
<i>A New Type of Number: Irrational Numbers</i>	59	Star Patterns with Geometric Division	73
The Ratio in a Square	59	<i>The Pentagon and the Golden Ratio</i>	74
Guessing the ratio	59	Constructing a Pentagon	74
The Great Pythagorean Crisis	59	Places where the Pentagon Appears	74
The Four Ratios of a Square	59	The Geometrical Properties of Nested Pentagons	75
Practice calculating the diagonal or the side	59	The Golden Ratio Φ	75
π - The Ratio in a Circle	59	The Fibonacci Sequence	75
The impossibility of measuring	59	The Golden Rectangle	76
Archimedes method for calculating π	59	The Rectangle of Whirling Squares	76
Decimal approximations for π	60	The Golden Triangle and its Spiral	76
Fractional approximations for π	60	<i>Angle Theorems and Proofs</i>	77
The Four Ratios of π .	60	Theorems from 2 Parallel Lines and a Transversal	77
Practice calculating the circumference or diameter	60	Corresponding angles are congruent	77
Repeating Decimals	61	Alternate interior angles are congruent	77
Two laws of repeating decimals	61	Same side interior angles add to 180°	77
Irrational Numbers	61	The Angles in a Triangle add to 180°	77
The Square Root Algorithm	61	Cutting Out Angles	77
		The Half-wheel Theorem	77

The Angles in Polygons other than Triangles	78	Density	95
Angle Puzzles	78	<i>Proportions</i>	97
Theorem of Thales	78	Shortcuts for Solving Proportions	97
Theorem of Morley	78	Moving along diagonals	97
<i>Pythagorean Theorem</i>	79	Cross multiplying	97
Visual Proofs	79	Word Problems that Use Proportions	97
A cut-out puzzle	79	Recipe problems	97
The case of the isosceles right triangle	79	Gas mileage problems	98
The case of the 3-4-5 triangle	79	Map scale problems	98
Pythagorean Triples	80	Rate problems	98
Pythagoras's formula	80	EIGHTH GRADE ALGEBRA	99
Plato's formula	80	<i>Expressions</i>	99
The Arabian formula	80	Order of Operations	99
The primitive Pythagorean Triples	80	Evaluating Expressions	99
Calculating Missing Sides of Triangles	80	The Laws of Exponents	100
<i>Other Topics</i>	81	Fractions and Negatives	100
EIGHTH GRADE ARITHMETIC	82	<i>Equations</i>	100
The year before high school	82	Use of Equal sign	100
The order of topics	82	Distributive Property	100
Main Lessons and Priorities	82	Equations with Fractions	101
<i>Number Bases</i>	82	Equations with Fractional Constants and Coefficients	101
Ancient Number Systems	83	Cross Multiplying	101
Expanded Notation	83	Strange Solutions	101
Scientific Notation	83	Equations with a solution of $X = 0$	101
Base-Eight, Octal	83	Equations where any value for X will work	101
Base-Five	84	Equations with no solution	101
Base-Sixteen, Hexadecimal	85	Converting Repeating Decimals into Fractions	102
Base-Two, Binary	86	EIGHTH GRADE COMPUTERS	103
Arithmetic in Various Bases	87	<i>Computer memory and ASCII code</i>	103
Multiplication tables	87	Binary Codes using Flags	103
The climax of the unit	88	A Computer Bit as a Switch	103
Multiplication with zeroes	88	One Byte of Memory	103
Converting between Binary and Hexadecimal	88	Decoding Strings of Binary Code	103
<i>The World of Numbers</i>	89	<i>Computer Algorithms</i>	104
The Square Root Algorithm without zeroes	89	Writing Familiar Algorithms	104
The Pythagorean Theorem	89	An algorithm for addition	104
The Hypotenuse Formula	89	An algorithm for long division	104
The Leg Formula	89	Examples of New Algorithms	104
Finding the missing sides of right triangles	89	The prime number algorithm	104
<i>Percents and Growth</i>	90	The square root algorithm – without zeroes	104
Calculators	90	EIGHTH GRADE GEOMETRY	105
Four Ways to Find the Base	90	<i>Mensuration (Areas and Volumes)</i>	105
The Even Multiple Method	90	Beware of Formulas!	105
The Decimal Method	90	The Pythagorean Theorem	105
The Fraction Method	90	Von Baravalle's proof	105
The Algebra Method	90	Area of a Trapezoid	106
Increase/Decrease Problems	90	Heron's Formula for the Area of a Triangle	106
Rewording a percent increase problem	90	Calculating the Area of Four Types of Triangles	106
Rewording a percent decrease problem	90	A right triangle	106
Calculating the Percentage of Increase or Decrease	90	An isosceles triangle	106
Review seventh grade	90	An equilateral triangle	106
Calculating the Starting Point	91	A scalene triangle	106
Exponential and Linear Growth	91	Area of a Circle	107
The Exponential Growth Formula	92	Proof of the formula $A = \pi \cdot r^2$	107
The Exponential Growth Table	92	Archimedes' version of the area of a circle	107
Dramatic results	92	Portions of Circles	107
Depreciation	92	Finding the length of an arc of a circle	107
The Rule of 72	93	Finding the area of a segment of a circle	107
<i>Dimensional Analysis</i>	93	The Basics of Volume	108
A Few Thoughts on this Unit	93	Cubic Measurement	108
Calculators	93	Notation	108
Conversion table	93	Don't give many formulas	108
Accuracy	93	Volumes of Prisms and Cylinders	108
Showing work	93	The transition from area to volume	108
Two Methods for doing Unit Conversion Problems	93	3-D shear and stretch for finding volumes	108
Using the Intuitive Approach	93	Volumes of Pyramids and Cones	109
Using the Chain Method	93	Archimedes' ratio	109
Converting between the U.S. and Metric System	94	The volume of a sphere $V = \frac{4}{3} \pi r^3$	110
Converting Units for Rates	94	5 HYPERLINK \l " _Toc388861961" Surface Area	110
Converting Units of speed	94	Surface area of a sphere	110
Unit cost	94	Surface area of a cone	110
Inverse ratios and reciprocals	95	Proof of the formula $S = \pi k r$	111
Converting Areas and Volumes	95		
Grains of rice problem	95		

Mensuration Practice Problems	111	Curves in Movement	130
A cylindrical can	111	A family of hyperbolas and ellipses	130
The volume of a sphere	111	Moving the focus outside the directrix	131
The volume of a cone	111	Turning the directrix circle inside-out	132
A triangular prism	111	The Curves of Cassini	133
Volume and surface area of a pyramid	112	What is a cassini curve?	133
A conical drinking glass	112	Formulas and set-up	133
The volume of an octahedron and tetrahedron	112	The transformation of a Cassini curve	134
Tricks with Dimensions	113	Construction of a Cassini Curve	135
Making a solid into a straight line	113	APPENDIX A – DRAWINGS	138
Are there too many people on the earth?	113	Equiangular Spirals	138
<i>Stereometry</i>	114	Rotations of Circles	139
Vocabulary	114	The Metamorphosis of a Limaçon	140
Types of Polyhedra	114	□ HYPERLINK \l " _Toc388862051" Hierarchy of	
The Platonic Solids	115	Knots and Interpenetrating polygons	142
The four properties	115	The King's Crown	143
Proof that there are only five Platonic solids	115	The 24-division with Diagonals	143
Plato's Academy	115	Star Patterns with Geometric Division	144
Kepler's universe	115	Theorem of Morley	145
The Transformation of Solids	116	The Perspective Reduction of a Figure	146
The transformation of solids in the mind	116	APPENDIX B – ADVANCED TOPICS	147
The transformation of solids using clay	116	Questions regarding Repeating Decimals	147
The evolution of solids	116	Lesson Plan for Square Root Algorithm	148
Transforming a cube into an octahedron	116	The Volume of an Octahedron and Tetrahedron	153
Transforming a cube into a tetrahedron	117	Proof that there exists only five Platonic solids	155
Transforming a cube into a dodecahedron	117	APPENDIX C – WONDER OF NUMBER	156
Transforming a cube into a rhombic dodecahedron	117	Square and Triangular Numbers	156
Transforming a dodecahedron into an icosahedron	117	Perfect, Abundant and Deficient Numbers	156
Pushing in the points of a tetrahedron	118	Sums and Differences Theorems	157
Orthogonal Views	118	Perfect and Abundant Numbers	159
Duality	118	Euclid's Formula for Perfect Numbers	160
Examples of dual solids	118	The First 100 Square Numbers	161
The Archimedean Solids	119	The First 75 Triangular Numbers	161
The Stretching Process	119	Powers of Two Table	162
The Archimedean Duals	120	Prime Numbers up to 2000	163
Constructing Paper Models	120	Even Numbers as the Sum of Two Primes	164
The possible nets for a cube and a tetrahedron	120	Odd Numbers as the Difference of Two Squares	165
Tips for constructing paper models	121	Numbers as the Sum of Two Squares	166
Drawing the net	121	APPENDIX D – TABLES & HANDOUTS	167
Putting it together	121	Sixth Grade Math Tricks	167
Close-Packing	122	Seventh Grade Math Tricks	168
Euler's Formula	122	Archimedean Solids and their Duals	169
Additional 3-D Transformation Exercises	122	Patterns for the Archimedean Duals	171
The inner-tube problem	122	π to 5000 Decimal Places	172
Reducing solids to tetrahedrons	122	Multiplication Tables for Number Bases	173
<i>Loci</i>	123	Place Value Table	173
Key Ideas	123	ASCII Code Table	174
What is Loci?	123	Binary/Hexadecimal Conversion Table	174
Why do we teach it?	123	An Algorithm for Addition	175
Loci as a main lesson	123	An Algorithm for Long Division	175
Requirements?	123	An Algorithm for Prime Numbers	176
The treasure hunt	123	The Square Root Algorithm - without zeroes	177
The process is important	123	Table of Squares	178
A Circle	123	Table of Square Roots	178
Two Parallel Lines	123	Growth Rate Table	179
Two Concentric Circles	123	Conversion Table	180
A Perpendicular Bisector	123	Fourth Grade Assessment Test	174
Two Angle Bisector	124	Sixth Grade Assessment Test	175
A Parabola	124	Summary of Math Skills	176
A parabola in movement	124	SUGGESTED READING	184
An Ellipse	126	SPECIAL SYMBOLS	185
An ellipse in movement	126	GLOSSARY	185
A Hyperbola	127	INDEX	189
A hyperbola in movement	127		
The two branches of a hyperbola	128		
Alternative Definitions	128		
The ellipse	128		
Proof of the locus definition of an ellipse	128		
The hyperbola	128		
Conic Sections	129		
What is a section?	129		
Conic sections in movement	129		
Conic sections from cones of light	130		

Thoughts on Teaching Math

What makes a good math student?

As teachers, we all hope our students will become good at math. But to realize this we need to fully understand what math is, and what it isn't (see *Blind Procedures*, above). So now we ask: What are the key attributes that enable a student to become good at math? Here is my short list:

- *Striving to understand deeply.* We want our students to understand the concepts they encounter. Good math students are never satisfied with going through a procedure without understanding what they are doing.
- *Asking good questions.* Good math students are curious, and wonder “what if...?” They question why something is true, and they become skillful at articulating questions.
- *Making mistakes.* Contrary to what many people think, mistakes are an important part of learning math. Good math students don't let mistakes discourage them. In fact, mistakes can motivate students to find the truth and make mathematical discoveries. We want our students to become comfortable with making mistakes, and to learn from their mistakes.
- *Attitude and work ethic.* This includes many things, such as: enthusiasm, determination, and discipline. Good math students persevere through their challenges; they are determined to succeed.

All of the above shows how our students learn many life lessons through studying mathematics. (And, yes, the same may be said about the proper teaching of other subjects as well.)

What makes a good math teacher?

Many class teachers feel under-confident in their own math skills, and, in some cases, have had traumatic experiences with math when they were in school. Often, this results in the teacher developing an antipathy towards math. However, if such a teacher can find a way to rise above his antipathy toward math, then that teacher may find joy in math, which can result in bringing wonderful math lessons to the students.

This is what I feel makes a good math teacher:

- *Enthusiasm for learning math.* For many teachers, this amounts to finding a new relationship to math. How wonderful it can be to find out that math can be interesting and rewarding!
- *Ability to present the material effectively.* This is the art of teaching.
- *Adequate preparation time for the math lessons.* With everything that is demanded of the class teacher, there often isn't enough time left to prepare adequately for the math lessons.
- *A healthy relationship to the students.* This helps to create a safe and comfortable learning environment.

Teaching the “Big Topics”

The “big topics” in middle school math are fractions, decimals, percents, ratios, and (simple) algebra. Our students should have these topics mastered before entering high school. There are two common mistakes made with these big topics.

The first is to do *too much too soon*. The topic may have been introduced in a wonderful and effective way, but if we build too much on the new foundation then many of the students may drown. The second mistake is *not enough follow-up and review*. This often happens with percents. It is introduced and practiced (perhaps too much!) in sixth grade, and then the students might never see it again.

So how should percents, for example, be done? It should be introduced in a wonderful sixth grade main lesson – not too much – and kept very simple. Then, one year later, the topic is reviewed and deepened – again being careful that it isn't too much. And then, once again, it is put to sleep. Now the stage has been set for going into depth in eighth grade. A similar three-step plan can be followed with any “big topic”. If we want the students to learn something well and permanently, then we need to create a “dance” between introducing, deepening, practicing, sleeping, and reviewing.

Separation of form and number

It is helpful to think of *form* (pure geometry) as having its roots in the physical/material world, and *number* as having its roots in the non-physical world of pure thought. In education today, *form* and *number* are often blended together. This can lead to unnecessary confusion.

There certainly are times when it is appropriate and helpful to integrate numbers into a geometry topic. For example, geometric figures become associated with numbers and algebraic formulas in the study of measurement (e.g., areas and volume). However, we also need to find ways to have our students experience “pure geometry” without attaching formulas and numbers. Waldorf schools do this starting in first grade with

Seventh Grade

The importance of seventh grade

Seventh grade is an important year academically. This is the year when students start to develop abstract thinking (through algebra, physics, essay writing, etc.). It is relatively common for a student to enter seventh grade fairly weak in math, but then to “wake-up” during seventh grade, and, in the end, to enter high school quite strong in math.

The order of topics

My seventh grade workbook (contact Jamie York Press for ordering) allows the students to practice their skills with most of the topics listed here, with a few exceptions (e.g., puzzle problems). The order of the units in my workbook is:

- | | | |
|----------------------|-------------------|---|
| 1. Arithmetic review | 4. Percents | 7. Geometry |
| 2. Measurement | 5. Ratios Part II | 8. Square Root Algorithm (optional) |
| 3. Ratios Part I | 6. Rates | 9. Algebra (for the <i>algebra</i> main lesson) |

Arithmetic

Review Sixth Grade

- Especially review fractions, decimals, and division (see 6th grade Arithmetic).
- Integrate review into new material, as feasible.

The World of Numbers

Math Tricks

- Review *sixth grade math tricks* (see Appendix D).
- Do the *seventh grade math tricks* (see Appendix D). Introduce perhaps one new trick each week, and work on practicing new ones with old ones during mental arithmetic. (See Introduction, *Mental arithmetic*.)

Divisibility Rules

- Review sixth grade *Divisibility Rules*, and then do these as well:
 - A number is evenly divisible by 6 only if it is divisible by both 2 and 3.
Example: 577,368 is evenly divisible by 6 because it is divisible by both 2 and 3 .
 - A number is evenly divisible by 8 only if the last 3 digits are divisible by 8. This is because it will evenly divide into any number of thousands.
Example: 8,736,104 is *not* evenly divisible by 8 because the last three digits aren’t divisible by 8.
 - A number is evenly divisible by 12 only if it is divisible by both 4 and 3.
Example: 57,481,932 is evenly divisible by 12 because it is divisible by both 4 and 3.
 - A number is evenly divisible by 11 only if the difference of the sums of every other digit is evenly divisible by 11.
Example: With 6,273,905, we get one sum by adding the digits 6, 7, 9, and 5 to get 27. The other sum comes from adding the digits 2, 3, and 0, which gives 5. The difference of the two sums is 27–5, which is 22. And since 22 is evenly divisible by 11, then we can say that the original number 6273905 is also evenly divisible by 11.
Example: With 378,543 both sums are equal to 15, making the difference equal to zero. Since zero is evenly divisible by 11, then we can say that 378543 is also evenly divisible by 11.
Example: With 68,479, the two sums are 19 and 15, which have a difference of 4. Therefore, we conclude that 68479 is not evenly divisible by 11.

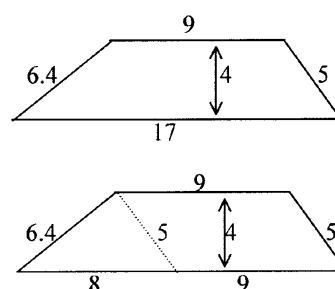
Area of a Trapezoid

- The formula for calculating the area of a trapezoid $A = \frac{1}{2}H(B_1 + B_2)$ is given in math textbooks, but I don't give it to the students. As a challenge problem, I often ask a student to come up with the formula.
- Have the students find the area of any trapezoid by dividing it into a triangle and a parallelogram (or a rectangle), or into two triangles.

Example: Find the area of the trapezoid shown on the right.

Assume all measurements are given in meters.

Solution: We first divide the trapezoid by drawing a line parallel to one side from one of the obtuse angles, as shown in the drawing at the right. We then calculate the area of the parallelogram as its base (9) times its height, which is 4 (not 5!), resulting in an area of 36m^2 . The triangle also has a height of 4, and its base is 8, so its area is $\frac{1}{2}(8)(4)$, which is 16m^2 . The whole trapezoid, therefore, has an area of $36+16 = 52\text{m}^2$.



Heron's Formula for the Area of a Triangle

Area $\Delta ABC = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$, where $s = \frac{1}{2}(a + b + c)$ is the semi-perimeter.

- This formula is attributed to the Greek, Heron (fl. $\approx 75\text{A.D.}$), but it may have been Archimedes that came up with it first.
- Heron's amazing proof of this formula is, for me, the climax of the tenth grade year of studying geometry.
- Before seeing this formula, the students should first be able to calculate the areas of non-right triangles where the base and height are given. (See 7th grade Geometry, Area.)
- The beauty of this little-known formula is that you don't need to know the height of the triangle. Without this formula, you would have to use trigonometry (studied in high school) to calculate the height, and it would be more complicated.

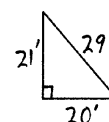
Example: Find the area of the triangle that has sides of length 5m, 6m, and 7m.

Solution: The perimeter is 18m, so the semi-perimeter is 9m. Putting all the numbers into the formula, we get: $\text{Area} = \sqrt{9(9-5)(9-6)(9-7)}$, which is $\sqrt{9 \cdot 4 \cdot 3 \cdot 2}$, and becomes $\sqrt{216}$. Using the square root algorithm, we get an area of 14.70m^2 (rounded).

Calculating the Area of Four Types of Triangles

- A right triangle.* We are given the base and the height, so finding the area is easy.

Example: With the triangle here, the area is: $A = \frac{1}{2} \cdot B \cdot H \rightarrow A = \frac{1}{2} \cdot 20 \cdot 21 \rightarrow A = 210\text{ft}^2$



- An isosceles triangle.* Here, we can use the Pythagorean Theorem in order to calculate the height. We then use this height in order to calculate the area.

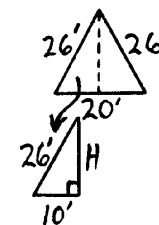
Example: We start with a triangle with one side 20' long and two sides 26' long. To find the height, we cut the triangle in half, which makes a right triangle with sides 26', 10', and H, which is the height of the original triangle. Using the leg formula we get:

$$H^2 = 26^2 - 10^2 \rightarrow H^2 = 676 - 100 \rightarrow H^2 = 576 \rightarrow H = 24$$

(We also could have determined H more quickly by using Pythagorean triples.)

Now we know that the height of the original triangle is 24. So the area is:

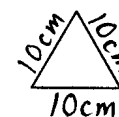
$$A = \frac{1}{2} \cdot B \cdot H \rightarrow A = \frac{1}{2} \cdot 20 \cdot 24 \rightarrow A = 240\text{ft}^2$$



- An equilateral triangle.* In this case, we could use the same method as described above for the isosceles triangle, but Heron's formula is generally easier.

Example: With an equilateral triangle that has all sides equal to 10cm, the perimeter is 30cm, so the semi-perimeter (S) is 15. The area of the triangle is then:

$$\text{Area} = \sqrt{15 \cdot (15-10) \cdot (15-10) \cdot (15-10)} \rightarrow \sqrt{15 \cdot 5 \cdot 5 \cdot 5} \rightarrow \sqrt{3 \cdot 5^2 \cdot 5^2} \rightarrow 5 \cdot 5 \cdot \sqrt{3} \rightarrow 25 \cdot (1.73) \approx 43.25\text{cm}^2$$



- A scalene triangle* (each side is different). In this case, we must use Heron's formula.

Example: Using Heron's formula with the triangle here, the perimeter is 72', so the semi-perimeter (S) is half of 72, which is 36. The area of the triangle is then:

$$\text{Area} = \sqrt{36 \cdot (36-28) \cdot (36-24) \cdot (36-20)} \rightarrow \sqrt{36 \cdot 8 \cdot 12 \cdot 16} \rightarrow \sqrt{55296} \approx 235.1\text{ft}^2$$

