Introduction to Calculus
Discovery Sheet #1

Series Formulas

Series Notation

1) Expand and then evaluate each series.

a) $\sum_{i=4}^{9} i$

b) $\sum_{i=0}^{4} 3i$

c) $\sum_{i=0}^{4} 3i+5$

d) $\sum_{i=1}^{3} \frac{1}{2^i}$

e) $\sum_{i=1}^{3} \frac{1}{i^2}$

f) $\sum_{i=1}^{4} 35+5i$

g) $\sum_{i=0}^{3} 40+5i$

h) $\sum_{i=8}^{11} 5i$

2) Rewrite the series in Σ notation, and then evaluate.

a) $5+10+15+20+25+30$

b) $7+10+13+16$

c) $1+2+3+\ldots+100$

Power Series Formula!

3) Multiply.

a) $(x-1)(x+1)$

b) $(x-1)(x^2+x+1)$

c) $(x-1)(x^3+x^2+x+1)$

d) $(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$

e) $(x-1)\sum_{i=0}^{6} x^i$

f) $(x-1)\sum_{i=0}^{22} x^i$

g) $(x-1)\sum_{i=0}^{n} x^i$

4) *Power Series Formula.* Use the previous problem to state a general formula for…

$\sum_{i=0}^{n} x^i$

5) Evaluate each expression:

a) $(1+3+3^2+3^3+3^4+3^5+3^6)$

b) $(1+2+2^2+\ldots+2^{18})$

c) $\sum_{i=0}^{6} 3^i$

d) $\sum_{i=0}^{18} 2^i$

Infinite Series

What happens if you add up infinitely many numbers? Most people would reply that you get an infinitely large result. But, what if the series had the special characteristic that each step got smaller? Let's see what happens…

6) Consider the following relation: $y = x^n$. Use a calculator to find y, given that…

a) $x=1.1$ and $n=10$

b) $x=1.1$ and $n=50$

c) $x=1.1$ and $n=500$

d) $x=1.1$ and $n=\infty$

e) $x=0.9$ and $n=10$

f) $x=0.9$ and $n=50$

g) $x=0.9$ and $n=500$

h) $x=0.9$ and $n=\infty$

7) What mathematical law is reflected in the above problems?

The Infinite Power Series Formula

The *Power Series Formula* (from above) states:

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$

8) If $0 < x < 1$, what is the value of x^{n+1} as n approaches infinity?

9) *Infinite Power Series Formula.* What does the *Power Series Formula* become when $0 < x < 1$ and n approaches infinity?

10) Evaluate each expression:

a) $1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$

b) The Dichotomy $\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$

c) $1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots$

d) $\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots$

e) $\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots$

Introduction to Calculus
Discovery Sheet #2

Limits & Average Speed

1) A cyclist is timed over the course of a 3-km ride. He starts his stopwatch as soon as he starts. He reaches the 800m marker at 61.9 seconds, reaches the 2000m marker at 161.4 seconds, and reaches the finish line (3000m) at 240.8 seconds.

a) What is the average speed of the cyclist from the first marker to the second marker?

b) What is the average speed of the cyclist from the second marker to the finish line?

c) What is the average speed of the cyclist from the first marker to the finish line?

d) What is the average speed of the cyclist throughout the whole race?

e) Give a formula for average speed given d_1, d_2, t_1, t_2.

2) What is your average speed…

a) if you go 6 miles in 2 hours?

b) if you go 6 miles in 1 hour?

c) if you go 6 miles in a ½ hour?

d) if you go 6 miles in a $\frac{1}{10}$ hour?

e) if you go 6 miles in a $\frac{1}{100}$ hour?

f) if you go 6 miles in a $\frac{1}{1000}$ hour?

g) if you go 6 miles in 0 hours?

h) if you go 0 miles in 6 hours?

i) if you go 0 miles in 0 hours?

3) Evaluate each limit. Use a calculator in order to find values very close to the given limit.

a) $\lim_{x \to 0} \frac{x^2+5x}{x}$

b) $\lim_{x \to 3} \frac{x^2-9}{x-3}$

c) $\lim_{x \to 3} \frac{x^2+9}{x-3}$

d) $\lim_{x \to 0} \frac{x^3-8x^2}{2x^2}$

e) $\lim_{x \to 5} \frac{x^2+9}{x-3}$

f) $\lim_{x \to 2} \frac{x^2+3x-10}{x-2}$

g) $\lim_{x \to \infty} \frac{6x^2-3x}{2x^2}$

h) Given $0 \le x < 1$ $\lim_{n \to \infty} x^n$

i) $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$

j) $\lim_{x \to \infty} \left(1+\frac{3}{x}\right)^x$

4) **Function Practice.**
Given these functions

$f(x) = 5x - 3$
$g(t) = \sqrt{t+2}$
$p(n) = n^2 + 5$

Evaluate each:

a) f(6)
b) g(23)
c) p(10)
d) g(47)
e) p(4)
f) f(x+2)
g) $p(n^4)$
h) p(t – 4)

Introduction to Calculus
Discovery Sheet #3

Galileo's Experiment

Galileo wanted to find out the relationship between distance and time when a body (e.g., a ball) is falling. Contrary to popular belief, he didn't drop balls from the Tower of Pisa. His timing instruments wouldn't have been accurate enough for something moving so fast. He needed to slow things down – so he rolled balls down inclined planes.

Although he may have done his experiments somewhat differently, we can imagine him collecting data by rolling a ball down a long inclined plane, and then marking the distance that the ball has traveled after each second has passed. (This makes time the control variable.) If the inclined plane has a 10° angle of inclination, we get the following data:

T (sec) →	0	1	2	3	4	5	6	7	8	9	10
D (meters) →	0	0.851	3.40	7.66	13.6	21.3	30.6	41.7	54.5	68.9	85.1

Questions:

1) What conclusions could Galileo have reached about the relationship of distance and time by looking at the above data? (Look for patterns!)

2) If a car is driving at a constant speed during an entire trip, then we can say that distance is directly proportional to time. What does this mean?

3) Fill in the Blanks, referring to the above inclined plane problem:

a) The distance is directly proportional to ______________.

b) The ratio of ________________ to _____________ is a constant.

c) The *constant of proportionality* is equal to ______________.

d) A formula that relates distance and time is _____

4) **Average Speed.**
Find the average speed of the ball…

a) in the first 10 seconds.

b) from 2 seconds to 7 seconds.

c) from 3 seconds to 8 seconds.

d) from 3 seconds to 5 seconds.

e) from 3 seconds to 4 seconds.

5) Find the instantaneous speed of the ball at 3 seconds.

6) **Function Practice.**
Given these functions

$$f(w) = 2w + 8$$
$$d(t) = 10 \cdot t^2$$
$$r(y) = y^2 + 2y - 1$$

Evaluate each:

a) f(5)

b) d(3)

c) r(10)

d) d(–2)

e) f(0)

f) r(3y)

g) f(5w)

h) f(w+5)

i) d(3t)

j) d(t+3)

Introduction to Calculus
Discovery Sheet #4

Instantaneous Speed, Part I

On the previous worksheet, one of the questions asked us to find the instantaneous speed at the instant that the ball had been rolling for exactly 3 seconds. The (seemingly insurmountable) difficulty with calculating instantaneous speed is that you need to divide the change of distance by the change of speed, and that would result in dividing zero by zero, which seems rather mind-boggling.

We will now work toward finding a general method for determining instantaneous speed.

1) If we adjust the steepness of the inclined plane to about 37.8°, we get a convenient distance formula:

$$d(t) = 3 \cdot t^2$$

Use this distance formula in order to calculate the average speed…

a) from 4 seconds to 6 seconds.
b) from 4 seconds to 5 seconds.
c) from 4 seconds to 4.5 seconds.
d) from 4 seconds to 4.1 seconds.
e) from 4 seconds to 4.01 seconds.
f) from 4 seconds to 4.001 seconds.
g) from 4 seconds to 4.0001 seconds.

2) Find the instantaneous speed at 4 seconds.

3) Now use the same method to find the instantaneous speed at 7 seconds.

4) Explain to one another how each of the following formulas represents average speed:

$$r = \frac{\Delta d}{\Delta t}$$

$$r = \frac{d_2 - d_1}{t_2 - t_1}$$

$$r = \frac{d(t_2) - d(t_1)}{t_2 - t_1}$$

$$r = \frac{d(t+h) - d(t)}{h}$$

"t" is the first time.
"h" is change in time.

5) Evaluate each limit. Use a calculator in order to find values very close to the limit.

a) $\lim\limits_{x \to -5} \frac{x^2 - 25}{x + 5}$

b) $\lim\limits_{x \to -5} \frac{x^2}{x + 5}$

c) $\lim\limits_{x \to 0} \frac{5x^2 - 12x}{x}$

d) $\lim\limits_{x \to 2} \frac{5x^2 - 12x}{x}$

e) $\lim\limits_{x \to \infty} \frac{8x^3 - 5}{2x^3}$

f) $\lim\limits_{x \to \infty} x^{1/x}$

g) $\lim\limits_{x \to 0} \frac{\sin x}{x}$

h) $\lim\limits_{x \to 0} \frac{\cos x - 1}{x}$

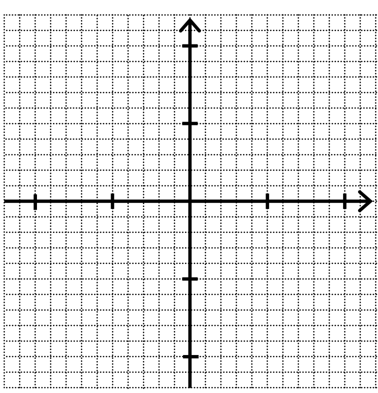
6) Graph each function

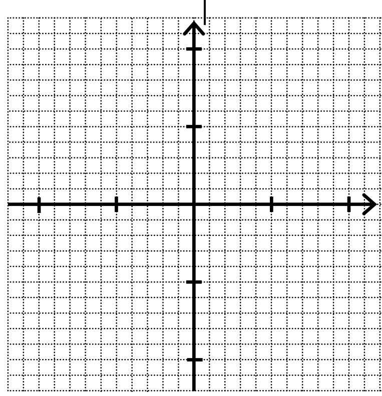
a) $f(x) = x^2 - 7$

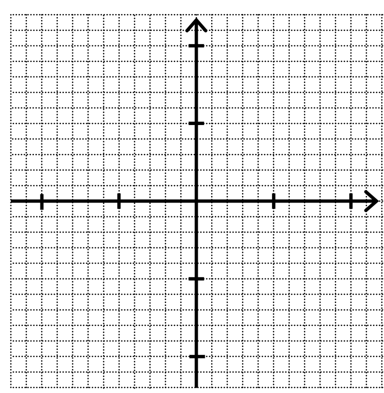
b) $f(x) = x^3$

c) $f(x) = \frac{1}{10} x^3$

d) $f(x) = -\frac{1}{10} x^3$







Introduction to Calculus
Discovery Sheet #5

Instantaneous Speed, Part II

1) Using the method from the previous worksheet, find the instantaneous speed…
   a) At 5 seconds, given that the distance formula is $d(t) = 3 \cdot t^2$.

   b) At 2 seconds, given that the distance formula is $d(t) = 3 \cdot t^2$.

   c) At 10 seconds, given that the distance formula is $d(t) = 3 \cdot t^2$

2) The *Calculus Average Speed Formula* is:

$$r = \frac{d(t+h) - d(t)}{h}$$

It is very important that we understand how to use this formula. It will help us to soon arrive at the concept of a derivative.

Use this formula (and show every step!) to do the below problems.

   a) Given $d(t) = 3 \cdot t^2$, find the *average speed* from 5 seconds to 8 seconds.

   b) Given $d(t) = 3 \cdot t^2$, find the *average speed* from 5 seconds to 12 seconds.

3) **New Formulas!** With each problem below, use the *Calculus Average Speed Formula* in order to derive a new formula. Be sure to show every step.
   a) Given $d(t) = 3 \cdot t^2$, give a formula for *average speed*, r, from 5 seconds to 5+h seconds.

   b) Given $d(t) = 3 \cdot t^2$, give a formula for *average speed*, r, from t seconds to t+h seconds.

   c) Given $d(t) = 3 \cdot t^2$, give a formula for *instantaneous speed*, v(t), at t seconds.

   d) Given $d(t) = k \cdot t^2$, give a formula for *instantaneous speed*, v(t), at t seconds.

4) Graph each function (on separate graph paper).
   a) $f(x) = -x^2 - 3x + 10$
   b) $f(x) = x^3 - 9x$
   c) $f(x) = -x^3 + 9x$
   d) $f(x) = x^3 + x^2 - 6x$

Introduction to Calculus
Discovery Sheet #6

The Derivative, Part I

Formulas for Instantaneous Speed!

On the previous sheet, we took a huge step. We found formulas for calculating average speed, r, and instantaneous speed, v(t). Why is this so important? Because we have managed to get around the paradox of instantaneous speed: dividing zero by zero!

These formulas (from the end of the previous sheet) are:

$r = 30 + 3h$

$r = 6t + 3h$

$v(t) = 6t$

$v(t) = 2kt$

1) Explain, once again, what each of the above formulas can be used for, and what each of the variables represents.

2) Use one of the above formulas to solve each.

a) Given $d(t) = 3 \cdot t^2$, find the *average speed* from 5 seconds to 12 seconds.

b) Given $d(t) = 3 \cdot t^2$, find the *average speed* from 3 seconds to 8 seconds.

c) Given $d(t) = 3 \cdot t^2$, find the *instantaneous speed* at 3 seconds.

d) Given $d(t) = 2.2 \cdot t^2$, find the *instantaneous speed* at 4 seconds.

e) Given $d(t) = 0.851 \cdot t^2$, find the *instantaneous speed* at 7 seconds.

f) Given $d(t) = 4.9 \cdot t^2$, find the *instantaneous speed* at 6 seconds.

3) Graph each function (on separate graph paper).

a) $f(x) = x^2 + 5x$

b) $f(x) = \frac{1}{4} x^3 + \frac{1}{2} x$

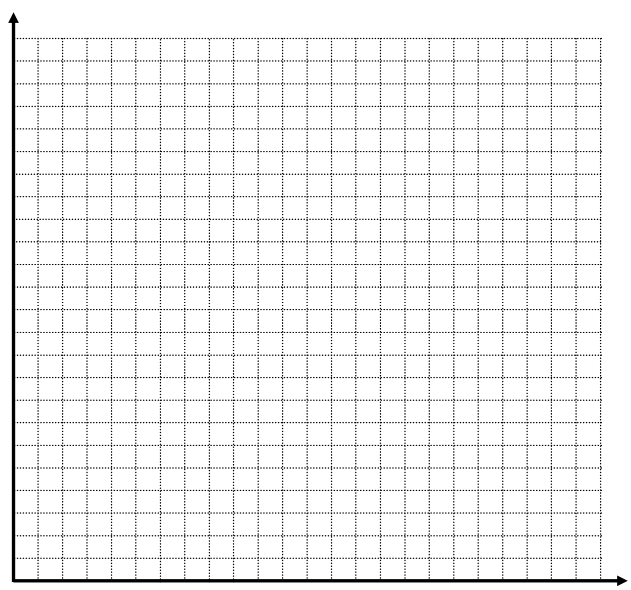
c) $f(x) = x^4 - 9x^2$

4) **Area Under a Curve**

a) Graph the equation $f(x) = x^2$. Shade in the area bounded by this curve, the x-axis, and the line x = 3.

b) Determine a method for approximating the area of this shaded-in region.

c) How can you make this approximation as accurate as desired?



Introduction to Calculus
Discovery Sheet #7

The Derivative, Part II

The Definition of the Derivative.

This is not really a formula. It simply tells us what to do in order to find a formula for the instantaneous rate of change for a given function, y = f(x).

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

1) Use the above definition of the derivative in order to determine the derivative of:

a) $f(x) = x^3$

b) $f(x) = x^2 + 7x$

2) Given what you have learned so far, what do you think the derivative of each function is?

a) $f(x) = x^7$

b) $f(x) = 4x^3$

c) $f(x) = x^5 + 7x^4 - 2x^3 + x^2 - 3x + 8$

3) *Derive the Exponent Law for Derivatives!*
Find $f'(x)$ given $f(x) = k\cdot x^n$

Summation Formulas

$$\sum_{i=1}^{n} i = \tfrac{1}{2}n^2 + \tfrac{1}{2}n$$

$$\sum_{i=1}^{n} i^2 = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n$$

$$\sum_{i=1}^{n} i^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 = \left(\sum_{i=1}^{n} i\right)^2$$

4) Use one of the above formulas in order to evaluate each series below.

a) $1 + 2 + 3 + \ldots + 30$

b) $1^2 + 2^2 + 3^2 + \ldots + 30^2$

c) $1^3 + 2^3 + 3^3 + \ldots + 100^3$

5) **Slopes of Lines!**
Find the slope of the line that…

a) passes through (5,7) and (3,2).

b) passes through (4,1) and (2,8).

c) passes through (–4,3) and (5,1).

d) passes through (36,23) and (86,43).

e) passes through (x_1,y_1) and (x_2,y_2).

Introduction to Calculus
Discovery Sheet #8

The Integral, Part I

Area Under a Curve (continued)

We will now continue working toward finding the area of the region bounded by the x-axis, the line x = 4, and the curve $f(x) = x^2$.

Our strategy is to approximate this area by drawing in a series of thin rectangles (each one with its top, right corner sitting on the curve), and adding up their areas.

We will arbitrarily assign n (the number of rectangles) to be 7, and hope that this gives us insight into the ultimate solution, which is achieved by having the number of rectangles go towards infinity.

1) Give the width of each rectangle (Δx).

2) What is the x-coordinate of the right side of the 5^{th} rectangle (x_5)?

3) What is the x-coordinate of the right side of the k^{th} rectangle (x_k)?

4) What is the height of the k^{th} rectangle?

5) What is the area of the k^{th} rectangle?

Now, we can say that the area of the region is approximately equal to the sum of the 7 rectangles, which would be:

$$Rect_1 + Rect_2 + Rect_3 + \ldots + Rect_7$$

$$1^2\left(\frac{4}{7}\right)^3 + 2^2\left(\frac{4}{7}\right)^3 + 3^2\left(\frac{4}{7}\right)^3 + \ldots + 7^2\left(\frac{4}{7}\right)^3$$

6) Find the sum of the areas of the 7 rectangles by further working the above sequence. (Hint: you'll need to use a summation formula.)

7) In order to derive a general formula, redo the entire above process, but with the number of rectangles equal to n, and the right boundary of the area set at x = a (instead of x = 4).

8) Use the above formula to answer the following questions:

a) What is the area with x = 4 as a right boundary, and using 500 rectangles?

b) What is the area with x = a as a right boundary, and using infinitely many rectangles?

The Integral

The formula found on the last problem is called the *Integral* of f(x). Using calculus notation, the process can be written as:

$$\int_0^a f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot\Delta x$$

(Where the width of the k^{th} rectangle is Δx, and the area of the k^{th} rectangle is $f(x_k)\cdot\Delta x$.)

We have just shown that for $f(x) = x^2$

$$\int_0^a f(x)dx = \frac{a^3}{3}$$

The left side should be read: "The integral of f(x) dx from x equals 0 to a", and it tells us the area under the curve from x = 0 to a.

9) Use the above integral formula to find the exact area under the curve $f(x) = x^2$…

a) from x = 0 to x = 4.

b) from x = 0 to x = 5.

c) from x = 0 to x = 9.

Derivative Practice

10) Given f(x), find the derivative, $\frac{dy}{dx}$.

a) $f(x) = x^4$

b) $f(x) = 6x^3$

c) $f(x) = 5x^6 + 3x$

d) $f(x) = 7$

e) $f(x) = 5x^4 - 8x^3 + x^2 + 9x - 7$

f) $f(x) = -x^8 + 3x^5 - 5x + 3$

Anti-Derivatives!

11) With each given f(x), find its anti-derivative, F(x), such that when you take the derivative of F(x) the result is f(x).

a) $f(x) = 7x^6$

b) $f(x) = 9$

c) $f(x) = 3x$

d) $f(x) = x^4$

e) $f(x) = 5x^3 + 8x^2$

Introduction to Calculus
Discovery Sheet #9

The Integral, Part II

The Integral

On the last worksheet, we were given the curve $f(x) = x^2$, and, through a long process of adding together infinitely many infinitely thin rectangles, we found a formula for calculating the area under the curve:

$$\int_0^a f(x)dx = \frac{a^3}{3}$$

This is called the *integral* of f(x), and is used to find the area under f(x) from x = 0 to a. For example, the area under the curve $f(x) = x^2$ from x = 0 to 6 is $\frac{1}{3}(6)^3 = 72$.

1) *A Short-Cut!* Fortunately there is a nice short-cut that enables us to derive the integral of a polynomial. We call this the *Power Rule for Integrals*.

a) What do you think the integral of $f(x) = 5x^2$ is?

b) Fill in the blank in order to state the *Power Rule for Integrals*.

If $f(x) = x^n$

Then $\int_0^a f(x)\,dx =$ _________

c) State in words, the meaning of *Power Rule for Integrals*.

2) With each problem below, first derive a general formula for calculating the area under the curve of f(x), then use it to calculate the requested area.

a) Given $f(x) = x^3$ calculate the area under the curve from x = 0 to x = 2.

b) Given $f(x) = \frac{1}{2}x^2$ calculate the area under the curve from x = 0 to x = 12.

c) Given $f(x) = 5x^4$ calculate the area under the curve from x = 0 to x = 3.

Anti-Derivatives

3) Given $f(x) = x^4$

a) Find $\int_0^a f(x)\,dx$

b) Find the anti-derivative, F(x).

c) What do the above answers tell you?

4) Given $f(x) = 8x^6$

a) Find $f'(x)$.

b) Find $\int_0^a f(x)\,dx$

c) Find the anti-derivative, F(x).

d) Find $F'(x)$.

Derivative Practice

5) a) Find $f'(x)$ given $f(x) = \frac{1}{4}x^2 + 8x - 7$

b) Calculate the instantaneous rate of change at x = 6.

6) a) Find $f'(x)$ given $f(x) = x^3 + \frac{1}{2}x^2$

b) Calculate the instantaneous rate of change at x = 2.

Comparing Graphs (*Challenge!*)

For each distance function given below, do the following:

- First, determine its derivative.
- Graph both the function and its derivative on the same graph.
- Lastly, look at the two graphs. Discuss what the graphs say about the movement of the object.

7) $d(t) = 2t$

8) $d(t) = \frac{1}{4}t^2$

9) $d(t) = -\frac{1}{2}t^2 + 3t$

10) $d(t) = t^3 - 6t^2 + 9t + 2$

Introduction to Calculus
Discovery Sheet #10

Tangents & Areas, Part I

** indicates more challenging, but important!

The Connection

We have seen that, for a given function, f(x), we can find a new function, the integral, that helps us find the area under the curve. We also saw how we could take the anti-derivative of f(x), which is F(x).

On the previous sheet, we came to the realization that the integral (which is geometric in nature) and the derivative (which is an algebraic process) are closely related; they are inverses of each other. Thus, the integral and the anti-derivative, F(x), are essentially equal.

What does all of this mean?

Quite simply, we can use F(x), the anti-derivative, to calculate integrals and find areas under curves.

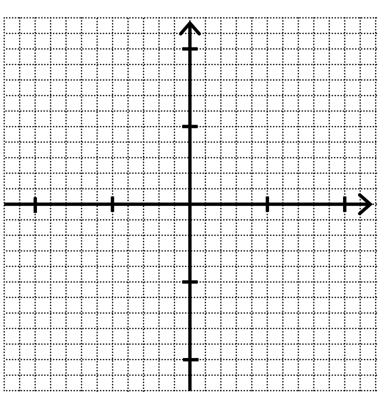
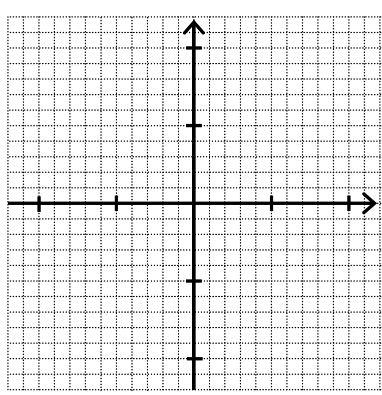
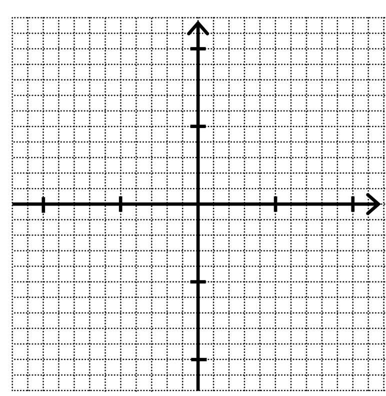
1) Calculate the area under the curve…
   a) $f(x) = x^3$ from x = 0 to 2.
   b) $f(x) = 7x + 5$ from x = 0 to 4.
   c) $f(x) = x^3 - 2x^2 + 3$ from x = 0 to 2.
   d) $f(x) = x^3 - 2x^2 + 3$ from x = 0 to 3.
   e) $f(x) = x^3 - 2x^2 + 3$ from x = 2 to 3.
   f) $f(x) = 5x^4$ from x = 3 to 5.

2) For any given function, f(x), what is $F'(x)$ always equal to?

Working with Curves

3) Given $f(x) = -x^2 + 8x - 12$…
   a) Graph f(x)
   b) Find $f'(x)$.
   c) What does $f(1) = -5$ mean?
   d) What does $f'(1) = 6$ mean?
   e) Find the slope of the curve at x = 3.
   f) Find the slope of the curve at x = 5.
   g) Give the points where f(x) = 0.
   h) Give the points where $f'(x) = 0$. (the local min and max coordinates)
   i) Calculate the area of the region bounded by the curve and the x-axis.

4) Given $f(x) = x^3 - 9x$…
   a) Make a table of values for f(x).
   b) Graph f(x) lightly in pencil.
   c) Determine $f'(x)$.
   d) Find $f(-2)$.
   e) Find $f'(-2)$. (Is this a surprise?)
   f) Give the points where f(x) = 0.
   g) **Give the points where $f'(x) = 0$. (the local min and max coordinates)
   h) Make improvements to your graph of f(x) so that it is very accurate.

Introduction to Calculus
Discovery Sheet #11

Tangents & Areas, Part II

More Graphing

1) Given $f(x) = -x^3 - 2x^2 + x + 2$…

a) Graph f(x)

b) Find f '(x).

c) Find f(2). What does this mean?

d) Find f '(2). What does this mean?

e) Where is x = 5?

f) Give the points where f(x) = 0.

g) **Give the points where f '(x) = 0.

h) Calculate the area of the region bounded by the curve that sits above the x-axis.

2) Given $f(x) = x^3 - 6x^2 + 11x$…

a) Determine f '(x).

b) Fill in the below table of values.

x	f(x)	f '(x)
4		
3		
2		
1		
0		
–1		

c) Give the points where f(x) = 0.

d) **Give the points where f '(x) = 0. (the local min and max coordinates)

e) Where on the graph is the slope = 74?

f) Make a very accurate graph of f(x).

Area Under a Curve

3) Find the area under the curve given…

a) $f(x) = 2x + 3$, $\int_2^4 f(x)\,dx$

b) $f(x) = -x^2 + 4x$, $\int_0^4 f(x)\,dx$

c) $f(x) = x^2 + x - 6$, $\int_{-2}^3 f(x)\,dx$

d) $f(x) = -x^3 - 2x^2 + x + 2$, $\int_{-1}^1 f(x)\,dx$

e) $f(x) = x^3 - 6x^2 + 11x$, $\int_1^3 f(x)\,dx$

Speed to Distance Problems (*Challenge!*)

4) The speed (in m/s) of an object is given by the formula $v(t) = 8t + 3$.

a) What is the formula for its distance?

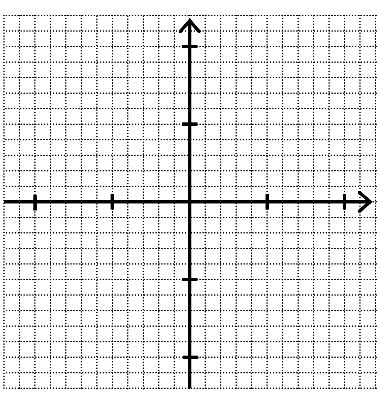
b) How far does the ball roll in 10 seconds?

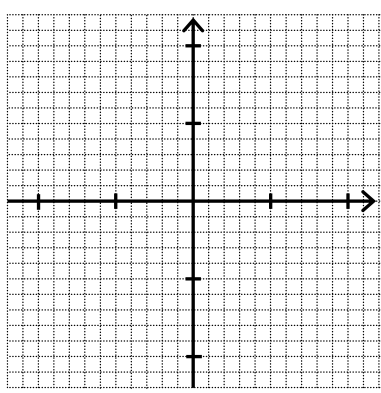
5) A ball, starting from stillness, rolls down an inclined plane. After 5 seconds, its speed is 23m/s.

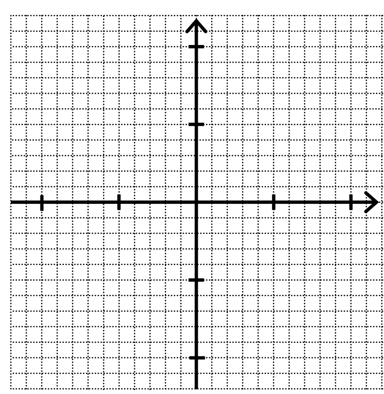
a) Find the formulas for distance and instantaneous speed?

b) What is the steepness of the inclined plane (in degrees)?

c) How far does the ball roll in 4 seconds?







Introduction to Calculus

Review Sheet

1) Given the distance formula $d(t) = 1.5 \cdot t^2 \ldots$

   a) Find the average speed from t = 4 to t = 6.

   b) Find the instantaneous speed at t=6.

   c) Find the average speed from t = 2 to t = 3.

   d) Find the instantaneous speed at t = 2.8.

2) Find the derivative, $f'(x)$, of…

   a) $f(x) = x^6$

   b) $f(x) = 10x^3$

   c) $f(x) = 8x$

   d) $f(x) = 5$

   e) $f(x) = 2x^5 - 3x^2 + 7$

3) Find the slope of the line tangent to the curve $f(x) = 2x^2 - 11$ at $x = 2$.

4) Find the anti-derivative, $F(x)$, of…

   a) $f(x) = x^6$

   b) $f(x) = 10x^3$

   c) $f(x) = 8x$

   d) $f(x) = 5$

   e) $f(x) = 2x^5 - 3x^2 + 7$

5) Find the area under the curve given…

   a) $f(x) = 3x^2$, $\int_2^5 f(x)\,dx$

   b) $f(x) = \frac{x}{3} + 6$, $\int_{-2}^5 f(x)\,dx$

   c) $f(x) = x^3 - 11x$, $\int_2^3 f(x)\,dx$

6) Given $f(x) = x^3 - 4x \ldots$

   a) Determine $f'(x)$.

   b) Fill in the below table of values.

x	f(x)	f '(x)
3		
2		
1		
0		
–1		
–2		
–3		

   c) Give the points where $f(x) = 0$.

   d) Give the points where $f'(x) = 0$. (the local min and max coordinates)

   e) Find the slope of the line tangent to the curve at $x = 3$.

   f) Make a very accurate graph of f(x).

   g) What does $f(1) = -3$ mean?

   h) What does $f'(1) = -1$ mean?

   i) What does $F(1) = -1\frac{3}{4}$ mean?

Topics to Study

- Average Speed and Instantaneous Speed
  - How are they different?
  - Why is instantaneous speed problematic?
- Definition of the Derivative
  - Explain what it means.
- The Integral and Area under the Curve
  - How is it that the integral is able to find the exact area under a curve?
- Fundamental Theorem of Calculus
  - What does it mean?
  - Why is it important?

Introduction to Calculus

Challenge Sheet!

On the Earth

The acceleration due to gravity of an object in free fall on earth is 9.8 m/s^2. This means that for every second, the speed increases by 9.8 m/s.

The acceleration (a) of a ball rolling down an inclined plane is $a = 9.8 \cdot \sin(\theta)$.

Once we know the acceleration (a), we can then easily derive a distance formula by using $d(t) = ½a \cdot t^2$. In the case of free fall, we then get $d(t) = 4.9 \cdot t^2$.

Example: If the inclined plane has a slope of 10°, then a = 1.702 m/s^2, and the distance formula is $d(t) = 0.851 \cdot t^2$ (which is what we used on sheet #2).

Example: If the inclined plane has a slope of 37.8°, then a = 6 m/s^2, and the distance formula is $d(t) = 3 \cdot t^2$ (which is what we used on sheet #3).

1) A ball is rolled down an inclined plane with a 45° inclination.
   a) Give the formula for the distance that the ball will travel after t seconds. (Assume no friction.)
   b) Give a formula for the instantaneous speed of the ball at t seconds.
   c) Give a formula for the instantaneous speed at the instant that the ball has rolled d meters.

2) A ramp is placed from the ground up to the top of a wall that is 4m above the ground. Find both the time it takes for a ball to roll down the ramp, and find its ending speed, given that the ramp…
   a) has an inclination of 15°.
   b) has an inclination of 30°.
   c) has an inclination of 60°.
   d) has an inclination of 90° (free fall).

On the Moon

The force of gravity is very nearly one-sixth as strong on the moon.

3) Redo problem #2a (above) as if it were on the moon.

4) Given your above answer, what is the ratio (moon:earth) of the times to get to the end of the ramp?

5) What is the ratio (moon:earth) of the speeds at the end of the ramp?

6) How long does it take to freefall 10m on earth, starting from rest?

7) How long does it take to freefall 10m on the moon, starting from rest?

8) What is the ratio of the above two answers?

9) Logan has an 80cm vertical leap (on earth).
   a) How long is she in the air?
   b) With what speed does she take off?
   c) What is her speed after 0.2 sec?
   d) What is her speed when she is 30cm off the ground?
   e) What would be her vertical leap on the moon? (Assume she leaves the ground with the same speed as on earth.) Find the ratio (moon:earth).
   f) How long would she be in the air on the moon?
   Find the ratio (moon:earth).

10) There are two Olympic ski jumps: one on the earth, and one on the moon. If a ball is rolled down each jump, and everything is identical about the two jumps, which ball will go further before landing?

Answers to Sheet #1

1) a) 39 b) 30 c) 55 d) $^7/_8$
e) $1\frac{13}{36}$ f) 190 g) 190 h) 190

2) a) $\sum_{i=1}^{6} 5i = 105$ b) $\sum_{i=1}^{4} 3i+4 = 46$ c) $\sum_{i=1}^{100} i = 5050$

3) a) $x^2 - 1$ b) $x^3 - 1$ c) $x^4 - 1$ d) $x^7 - 1$
e) $x^7 - 1$ f) $x^{23} - 1$ g) $x^{n+1} - 1$

4) $\frac{x^{n+1} - 1}{x - 1}$

5) a) 1093 b) 524,287 c) 1093 d) 524,287

6) a) ≈ 2.59 b) ≈ 117 c) $\approx 4.97 \cdot 10^{20}$ d) ∞
e) ≈ 0.349 f) ≈ 0.00515 g) $\approx 1.32 \cdot 10^{-23}$ h) 0

7) If $0 \le x < 1$ then $\lim_{x \to \infty} x^n = 0$

8) 0

9) If $0 \le x < 1$ then $\sum_{i=0}^{\infty} x^i = \frac{-1}{x-1} = \frac{1}{1-x}$

10) a) 2 b) 1 c) $^3/_2$ d) ½ e) 3

Answers to Sheet #2

1) a) $\approx 12.06\ ^m/_s$ b) $12.59\ ^m/_s$ c) $12.30\ ^m/_s$
d) $12.46\ ^m/_s$ e) $r = \frac{d_2 - d_1}{t_2 - t_1}$

2) a) 3 mph b) 6 mph c) 12 mph d) 60 mph
e) 600 mph f) 6000 mph g) ∞ h) 0 mph i) ?

3) a) 5 b) 6 c) no limit d) -4 e) 17
f) 7 g) 3 h) 0 i) $e \approx 2.7183$ j) $e^3 \approx 20.08$

4) a) 27 b) 5 c) 105 d) 7 e) 21
f) $5x + 7$ g) $n^8 + 5$ h) $t^2 - 8t + 21$

Answers to Sheet #3

2) If the time goes up, then the distance goes up by the same proportion.

3) a) the square of the time
b) distance to time squared
c) 0.851
d) $d(t) = 0.851\, t^2$

4) a) $8.51\ ^m/_s$ b) 7.66 c) 9.37 d) 6.82 e) 5.94

5) ??

6) a) 18 b) 90 c) 119 d) 40 e) 8
f) $9y^2 + 6y - 1$ g) $10w + 8$ i) $90t^2$
j) $10t^2 + 60t + 90$

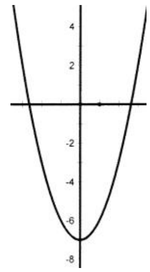
Answers to Sheet #4

1) a) $30\ ^m/_s$ b) $27\ ^m/_s$ c) $25.5\ ^m/_s$ d) $24.3\ ^m/_s$
e) $24.03\ ^m/_s$ f) $24.003\ ^m/_s$ g) $24.0003\ ^m/_s$

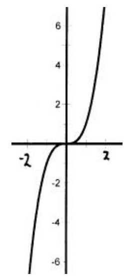
2) $24\ ^m/_s$ 3) $42\ ^m/_s$

5) a) -10 b) no limit c) -12 d) -2
e) 4 f) 1 g) 1 h) 0

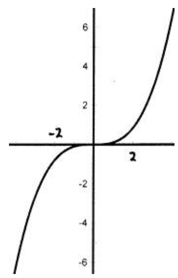
6) a)



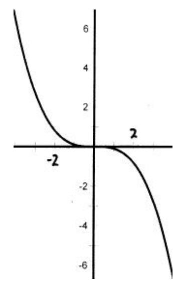
b)



c)



d)



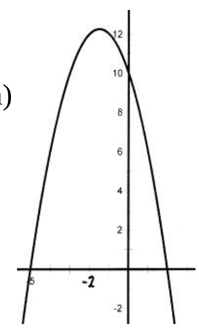
Answers to Sheet #5

1) a) $30\ ^m/_s$ b) $12\ ^m/_s$ c) $60\ ^m/_s$

2) a) $39\ ^m/_s$ b) $51\ ^m/_s$

3) a) $r = 30 + 3h$ b) $r = 6t + 3h$
c) $v(t) = 6t$ d) $v(t) = 2\,k\,t$

4) a)



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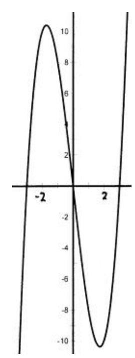
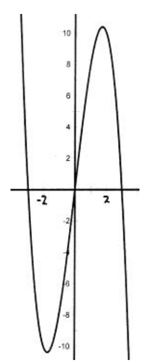
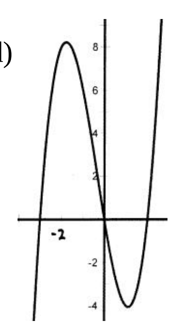


c)

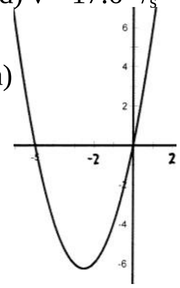


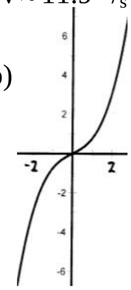
d)

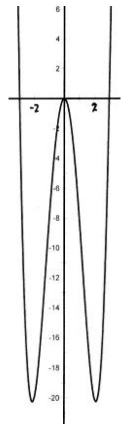


Answers to Sheet #6

2) a) $r = 51\ ^{m}/_{s}$ b) $r = 33\ ^{m}/_{s}$ c) $v = 18\ ^{m}/_{s}$
d) $v = 17.6\ ^{m}/_{s}$ e) $v \approx 11.9\ ^{m}/_{s}$ f) $v = 58.8\ ^{m}/_{s}$

3) a) 

b) 

c) 

Answers to Sheet #7

1) a) $f(x) = x^3$ We will need f(x+h)

$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

Using the definition to find the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - (x^3)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$f'(x) = \lim_{h\to 0} 3x^2 + 3xh + h^2$$

Now we can plug in 0 for h, which gives us

Our desired answer: $\mathbf{f'(x) = 3x^2}$

1) b) $f(x) = x^2 + 7x$ We will need f(x+h)

$f(x+h) = (x+h)^2 + 7(x+h)$
$= x^2 + 2xh + h^2 + 7x + 7h$

Using the definition to find the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 + 7x + 7h) - (x^2 + 7x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2xh + h^2 + 7h}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{h(2x + h + 7)}{h}$$

$$f'(x) = \lim_{h\to 0} 2x + h + 7$$

Now we can plug in 0 for h, which gives us

Our desired answer: $\mathbf{f'(x) = 2x + 7}$

2) a) $7x^6$ b) $12x^2$ c) $5x^4 + 28x^3 - 6x^2 + 2x - 3$

3) $f'(x) = k \cdot n \cdot x^{n-1}$

4) a) 465 b) 9455 c) 25,502,500

5) a) $^5/_2$ b) $-^7/_2$ c) $-^2/_9$ d) $^2/_5$ e) $\frac{y_2 - y_1}{x_2 - x_1}$

Answers to Sheet #8

1) $\frac{4}{7}$ 2) $5 \cdot \frac{4}{7} \to \frac{20}{7}$ 3) $k \cdot \frac{4}{7} \to \frac{4k}{7}$

4) $\frac{k^2 4^2}{7^2}$ 5) $\frac{k^2 4^3}{7^3}$

6) $1^2\left(\frac{4}{7}\right)^3 + 2^2\left(\frac{4}{7}\right)^3 + 3^2\left(\frac{4}{7}\right)^3 + \ldots + 7^2\left(\frac{4}{7}\right)^3$

$= \frac{4^3}{7^3}\left(1^2 + 2^2 + 3^2 + \ldots + 7^2\right) = \sum_{i=1}^{7} i^2$

Now we use the summation formula:

$\sum_{i=1}^{n} i^2 = ⅓\, n^3 + ½\, n^2 + {}^1/_6\, n$, which gives us:

$\frac{4^3}{7^3}\left(⅓\, 7^3 + ½\, 7^2 + {}^1/_6\, 7\right) \longrightarrow \frac{4^3}{3} + \frac{4^3}{2\cdot 7} + \frac{4^3}{6\cdot 7^2}$

≈ 26.12

7) Following the same process, we get:
$\frac{a^3}{n^3}\left(⅓\, n^3 + ½\, n^2 + {}^1/_6\, n\right) \longrightarrow \frac{a^3}{3} + \frac{a^3}{2\cdot n} + \frac{a^3}{6\cdot n^2}$

8) a) ≈ 21.40
b) as n approaches infinity, the second two factional terms approach zero. Therefore the answer is $\frac{a^3}{3}$.

9) a) 21⅓ b) 41⅔ c) 243

10) a) $4x^3$ b) $18x^2$ c) $30x^5 + 3$ d) 0
e) $20x^3 - 24x^2 + 2x + 9$ f) $-8x^7 + 15x^4 - 5$

11) a) $F(x) = x^7 + C$
b) $F(x) = 9x + C$
c) $F(x) = \frac{3x^2}{2} + C$
d) $F(x) = \frac{x^5}{5} + C$
e) $F(x) = \frac{5x^4}{4} + \frac{8x^3}{3} + C$

Answers to Sheet #9

1) a) $\frac{5a^3}{3}$ b) $\int_0^a f(x)\,dx = \frac{a^{n+1}}{n+1}$

2) a) $\int_0^a f(x)\,dx = \frac{a^4}{4}$; $a = 4 \to$ area = 4

b) $\int_0^a f(x)\,dx = \frac{a^3}{6}$; $a = 12 \to$ area = 288

c) $\int_0^a f(x)\,dx = a^5$; $a = 3 \to$ area = 243

3) a) $\frac{a^5}{5}$ b) $\frac{x^5}{5} + C$

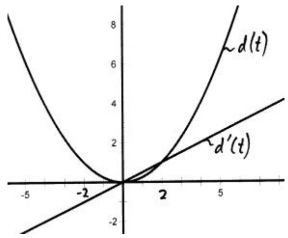
4) a) $48x^5$ b) $\frac{8a^7}{7}$ c) $\frac{8x^7}{7}+C$ d) $8x^6$

5) a) $f'(x) = \frac{1}{2}x + 8$ b) $f'(6) = 11$

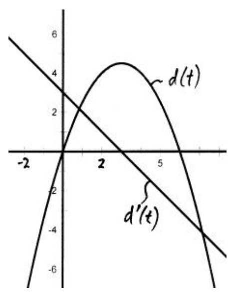
6) a) $f'(x) = 3x^2 + x$ b) $f'(2) = 14$

7) $d'(t) = v(t) = 2$
The object is moving at a constant speed

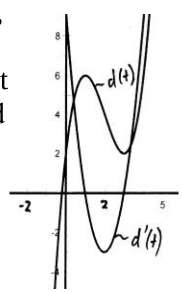
8) $d'(t) = v(t) = \frac{1}{2}t$
One possibility is to imagine a car moving toward a wall. "d" is the distance away from the wall. The car steadily decelerates, touches the wall, and then goes backwards with steady acceleration.



9) $d'(t) = v(t) = -t + 3$
One possibility is that the object is thrown up an inclined plane, and "d" is the object's distance uphill from a mark on the plane. Note that the maximum height (above the mark) is when $t = 3$, which is also when the height is zero.



10) $d'(t) = v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$
As opposed to the above problem, this case does not have constant acceleration. We can imagine that "d" is the height above the ground of a rocket. There are two moments when the rocket stops moving up (or down), and that's when its speed is zero.



Answers to Sheet #10

1) a) $F(x) = \frac{x^4}{4}$; $F(2) = 4$

b) $F(x) = \frac{7x^2}{2} + 5x$; $F(4) = 76$

c) $F(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x$; $F(2) = 4\frac{2}{3}$

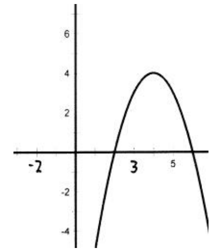
d) Same as above; $F(3) = 11\frac{1}{4}$

e) $F(3) - F(2) = 11\frac{1}{4} - 4\frac{2}{3} = 6\frac{7}{12}$

f) $F(x) = \frac{x^5}{5}$; $F(5) - F(3) = 3125 - 243 = 2882$

2) $F'(x) = f(x)$

3) a) See graph at right



b) $f'(x) = -2x + 8$

c) When $x = 1$, the y-value is -5

d) When $x = 1$, the slope of the curve is 6

e) $f'(3) = 2$

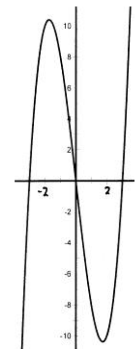
f) $f'(5) = -2$

g) $0 = -x^2 + 8x - 12 \rightarrow (2,0); (6,0)$

h) $0 = -2x + 8 \rightarrow x = 4$; $f(4) = 4$
The local max (slope = 0) is (4,4)

i) $F(x) = -\frac{x^3}{3} + 4x^2 - 12x$; roots are 2 and 6
$F(6) - F(2) = 0 - (-10\frac{2}{3})$; Area = $10\frac{2}{3}$

4) c) $f'(x) = 3x^2 - 9$ d) 10 e) 3



f) $f(x) = 0$ gives the roots:
(-3,0); (0,0); (3,0)

g) $0 = 3x^2 - 9 \rightarrow x^2 = 3$
$x = \pm\sqrt{3} \approx \pm 1.73$ To find the y-values, we plug into f(x).
$f(\sqrt{3}) \approx -10.4$; $f(-\sqrt{3}) \approx 10.4$
Max & min at (1.73, -10.4); (-1.73, 10.4)

Answers to Sheet #11

1) Given $f(x) = -x^3 - 2x^2 + x + 2\ldots$

a) See graph on next page with #2.

b) $f'(x) = -3x^2 - 4x + 1$

c) $f(2) = -12$. $(2,-12)$ is a point on the curve.

d) $f'(2) = -19$. At $x = 2$, the slope is -19.

e) $y = -168$ when $x = 5$. It is the point $(5,-168)$

f) $y = 0$ at $(-2,0)$; $(-1,0)$; $(1,0)$.

g) We set the derivative equal to zero, and then use the quadratic formula.
Ans: (0.215, 2.11); (–1.55, –0.631)

h) $\int_{-1}^{1} f(x)\,dx = F(1) - F(-1)$

$F(x) = -\frac{x^4}{4} - 2\frac{x^3}{3} + \frac{x^2}{2} + 2x$

$F(1) = -\frac{1}{4} - \frac{2}{3} + \frac{1}{2} + 2 = 1\frac{7}{12}$

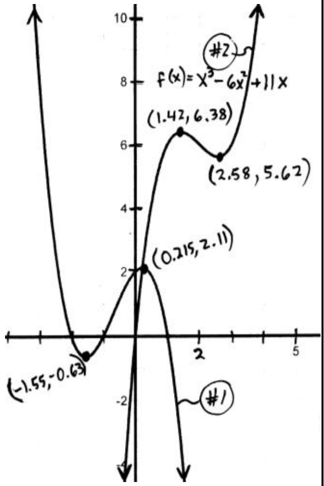
$F(-1) = -\frac{1}{4} + \frac{2}{3} + \frac{1}{2} - 2 = -1\frac{1}{12}$

Area = $F(1) - F(-1) = 1\frac{7}{12} - (-1\frac{1}{12}) = \mathbf{2\frac{2}{3}}$

2) a) $f'(x) = 3x^2 - 12x + 11$

b)

x	f(x)	f '(x)
4	12	11
3	6	2
2	6	-1
1	6	2
0	0	11
-1	-18	26



c) The only place it crosses the x-axis is (0,0).

d) Max and Min are (2.58, 5.62); (1.42, 6.38)

e) We solve $74 = 3x^2 - 12x + 11$ to get $x = -3, 7$. Ans: (–3, –114) and (7,126)

3) a) $F(x) = x^2 + 3x$; Area = F(4) – F(2) = **<u>18</u>**

b) F(4) = **<u>10⅔</u>** c) –4½ – 11⅓ = **–15 $\frac{5}{6}$**

d) $1\frac{7}{12} - (-1\frac{1}{12})$ = **<u>2⅔</u>** e) 15¾ – 3¾ = **<u>12</u>**

4) a) $d(t) = 4t^2 + 3t$
b) 430m

5) a) $v(t) = 4.6t$ and $d(t) = 2.3t^2$
b) $\sin^{-1}(2.3 \div 4.9) \approx 28.0°$
c) 36.8m

Answers to Review Sheet

1) a) Using one of the average speed formulas:

$$r = \frac{d(6) - d(4)}{6 - 4} \rightarrow \frac{54 - 24}{6 - 4} \rightarrow \frac{30}{2} \rightarrow \mathbf{\underline{15 m/s}}$$

b) Here are three ways to do this:

- Plug in two time values *very* close to 6.
- Plug in t = 6 and t = 6+h into the calculus average speed formula, then, after the H cancels, put zero in for h.
- Recall that the instantaneous speed formula for an inclined plane is v(t) = 2kt. In this case k = 1.5. Therefore v(t) = 3t

Each method yields the answer: **<u>18m/s</u>**

1) c) 7.5 m/s d) 8.4 m/s

2) a) $f'(x) = 6x^5$ b) $f'(x) = 30x^2$ c) f '(x) = 8
d) f '(x) = 0 e) $f'(x) = 10x^4 - 6x$

3) f '(x) = 4x ; f '(2) = 4•2 = **<u>8</u>**

4) a) $F(x) = \frac{x^7}{7} + C$ b) $F(x) = \frac{5x^4}{2} + C$ c) $F(x) = 4x^2 + C$

d) F(x) = 5x + C e) $F(x) = \frac{x^6}{3} - x^3 + 7x + C$

5) a) $F(x) = x^3 \rightarrow F(5) - F(2) \rightarrow 125 - 8 \rightarrow$ **<u>117</u>**

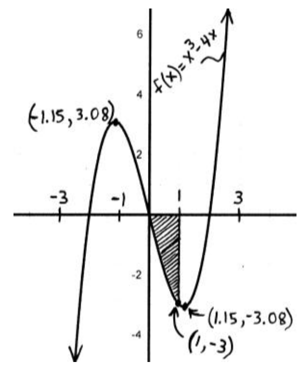
b) $F(x) = \frac{x^2}{6} + 6x \rightarrow 34\frac{1}{6} - (-11⅓) \rightarrow$ **<u>45½</u>**

c) $F(x) = \frac{x^4}{4} - \frac{11x^2}{2} \rightarrow$ –29¼ – (–18) $\rightarrow$ **<u>–11¼</u>**

6) a) $f'(x) = 3x^2 - 4$

b)

x	f(x)	f '(x)
3	15	23
2	0	8
1	–3	–1
0	0	–4
–1	3	–1
–2	0	8
–3	–15	23



c) It crosses the x-axis at **<u>(0,0); (2,0); (–2,0)</u>**.

d) $f'(x) = 0 = 3x^2 - 4 \rightarrow x = \pm 1.15$
the y-coordinates are $f(\pm 1.15) = \mp 3.08$
Therefore the Max/Min points are:
<u>(1.15, –3.08); (–1.15, 3.08)</u>

e) The slope at x = 3 is f '(3) $\rightarrow$ **<u>23</u>**

f) See graph above.

g) At x = 1 the y-coordinate is –3.

h) At x = 1 the slope is –1.

i) The area (shaded) from x = 0 to 1 is –1¾.

Answers to Challenge Sheet

1)a) $d(t) = 3.46t^2$ b) $v(t) = 6.93t$ c) $v(d) = 3.72\sqrt{d}$

2) a) $v(t) = 2.536t$; $d(t) = 1.268t^2$
d = 15.45m ; t = 3.49 sec ; v = 8.86 m/s
b) t = 1.81 sec ; v = 8.86 m/s
c) t = 1.04 sec ; v = 8.86 m/s
d) t = 0.904 sec ; v = 8.86 m/s

3) t = 8.53 sec ; v = 3.618 m/s

4) moon : earth = $\sqrt{6}$: 1

5) moon : earth = 1 : $\sqrt{6}$ or $\sqrt{6}$: 6

6) 1.43 sec

7) 3.50 sec

8) moon : earth = $\sqrt{6}$: 1

9) a) 0.808 sec b) 3.96 m/s c) 2.0 m/s

d) $v(t) = 3.96 - 9.8t \rightarrow d(t) = 3.96t - 4.9t^2$
$0.3 = 3.96t - 4.9t^2 \rightarrow t = 0.0846$
v(0.0846) = 3.13 m/s

e) 6:1 ratio $\rightarrow$ 4.8 m

f) 4.848 sec