Lesson Plans for

The Geometry and Philosophy of René Descartes

11th Grade Main Lesson (last updated Sept 2016)

Overview

This block is designed to be the students' first introduction to coordinate geometry. It is best if the students have had no exposure to coordinate geometry prior to this block. This is a radically different approach from the norm, where much graphing is done in 9th grade and plotting points is often introduced as early as 4th grade. However, without a proper context, this can be quite meaningless for the students.

The objective in this main lesson is to give a historical basis so that the students realize the profound significance of how Descartes' new geometry was able to unite the previous separate fields of algebra and geometry. If this is done well, then the students can really appreciate that coordinate geometry is a powerful mathematical tool, and they learn about how Descartes' new method was the first step on a new course for humanity. Much of this block centers around a study of Descartes' book (*Discours de la Méthode*), in which Descartes first publishes (in 1637) his new ideas on geometry and science. The students read the original text in order to gain a real understanding of Descartes' accomplishments.

This course is, to a large degree, a history of the evolution of thinking and human consciousness.

It is best for this course to be the first main lesson of the year, thereby allowing the 11th grade math track class to thoroughly cover traditional Cartesian Geometry through the remainder of the year.

Day #1

Questions for discussion: (First in groups, then share thoughts in class discussion.)

- *How Big is the Sun?* One day in the winter of Shanti's third grade year at Shining Mountain Waldorf School, she is talking with her younger brother, Arie, who is in kindergarten, about the sun. Shanti boasts that she is studying the sun in school. She tells of how God had gathered the light in order to create the sun. Arie then asks his sister, in order to put her knowledge to the test, "So, how big is the sun?" Shanti responds vaguely that it is really, really big. Arie is unsatisfied and asks for more precision. Shanti thinks for a bit as they both watch the sun set behind the house across the street. Then Shanti states, rather dramatically, "It's bigger than that house!" This greatly impresses Arie.
- *How do you know how big* (864,000 miles in diameter) *the sun is*? The sun and the moon appear to be about the same size in the sky, and yet the sun's diameter is about 100 times that of the Earth and the moon's diameter is about ¼ of the Earth's.
- *How did the Universe Begin?* If might be good to begin by asking the class this question to see how much (they think) they know.

Central Questions. This all leads to the following key questions (leave these questions hanging at least until tomorrow.)

- How do you know what you know?
- How do you know anything for certain?
- Throughout history, what individuals have shaped the course of humanity?

The Rectangle and Square Puzzle (Only do part a on the opening day of the main lesson):

Find the height of the rectangle that has a length of 10 cm and an area equal to the sum of the areas of two squares – one square having the same height as the rectangle, and the other having an area of:

(a) 16 cm^2. (b) 15 cm^2. (c) g^2, and the length of the rectangle is f instead of 10cm.

A Modern Solution: Have the students work *in groups* to find a solution. Invariably, they draw a diagram, label the height of both the rectangle and the unknown square as x, and then they set up an equation and solve it algebraically. Using this method, they get the following answers:

(a) $10x = x^2 + 16 \quad \rightarrow \quad x = 2 \text{ or } 8$

(b) $10x = x^2 + 15 \quad \rightarrow \quad x = 5 \pm \sqrt{10}$

(c) $fx = x^2 + g^2 \quad \rightarrow \quad x = \frac{f \pm \sqrt{f^2 - 4g^2}}{2}$

Descartes got the equivalent $z = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b^2}$ (using a, b and z instead of f, g and x)

Day #2

Discussion: Continue conversation from yesterday, in this order:

- What do we know about how big the sun is, and about how the universe began?
- How do we know these things?
- Go over hanging questions from yesterday:
  - How do you know what you know?
  - How do you know anything for certain?

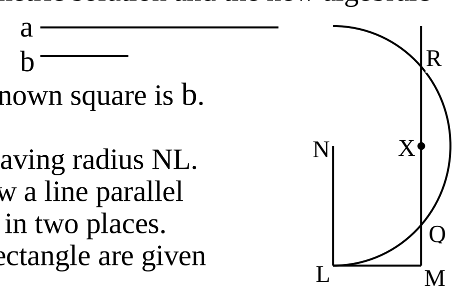
- In groups, have the students do parts b and c of the Square & Rectangle problem.
- Mention Descartes' book.
- In order to be able to fully comprehend the significance of Descartes' accomplishments, we have to be able to live into the time that he lived. Descartes' position in history:
  - Power is shifting from Southern Europe to Northern Europe – Spanish Armada is 1588.
  - Trade is starting to pick up, but society is still largely agrarian. Factories are still non-existent.
  - Cities are getting crowded. In 1600, London's population is 200,000; in 1700 it is 700,000.
  - Many illnesses are common; infant mortality is high; and terrible plagues come every 100 years or so (1344-1351 kills ¼ to ⅓ of Europe; also 1550 and 1665).
  - The reformation (16^{th} century) has divided Christianity.
    - Guttenburg's printing of the Bible was 1455
    - Martin Luther dies in 1546 and Henry VIII (England) dies in 1547.
    - The Inquisition is in full swing with Galileo being tried in 1633.
  - France. Henri IV 1589-1610. Louis XIII 1610-1643. Louis XIV 1643-1715 (72-year reign).
  - England. 1616: Shakespeare dies. 1603: Elizabeth I (Henry VIII's daughter) dies 45 year rule.
  - Descartes was 32 years younger than Galileo (trial in 1633), and 25 years younger than Kepler. He was almost the same age as Desargues and Fermat. He was 27 years older than Pascal, 47 years older than Newton and 50 years older than Leibniz.
  - Descartes lived at the beginning of the 17^{th} century "mathematical revolution". It went from algebra being developed as a language to the development of calculus by Newton and Leibniz.
- In groups (and then sharing with the full class) answer these questions:
  - How would Descartes' time be different from today, in terms of…
    - People's world view (i.e., how they look at and understood the world)?
    - The basis for people's thinking?
- Hanging Question: What do you know about evolution?

Day #3

- Continuing Yesterday's discussion:
  - Given what we concluded yesterday about "how do you know anything for certain?" (e.g., One class concluded that we can only be certain through observation/experimentation followed by a logical process leading to sound conclusions) – does this therefore mean that only physical things can exist or be true? Does it eliminate the possibility of the existence of God or anything spiritual?
  - Review (briefly) the idea of how people's thinking was different then.
- Evolution:
  - Misconception: Humans are not descended from apes, but we have a common ancestor.
  - No individual can change and pass on a trait.

- Explain what *Natural Selection* means.
  - Jean-Baptiste Lamarck (ca. 1800) put forth the idea that an organism can pass on characteristics that it acquired during its lifetime to its offspring. Give the example of the giraff. Lamarck's theory of evolution is called "inheritance of acquired characteristics" and generally isn't accepted today.
    - He also said that a specie tends to evolve toward increasing complexity.
    - He believed in sponteaneous generation.
  - Natural Selection states that individuals which are more adept at survival in a given environment, due to their genetic variation, will pass on their genes at higher rates within a species.
- Darwin's evolution should not be viewed as "Survival of the Fittest", but rather "Survival of the Adequate".
- Darwin's trip on *The Beagle* was 1831-1836. He was finished with *Origin* in 1844, but waited until 1859 to publish it.
- <u>What is missing from this picture of the evolution of the human being</u>? This conversation should lead to the idea that not only has the human body evolved over time, but so has thinking, or consciousness.

- Review briefly the rectangle/square problem.
  - Discussion of the rectangle/square problem:
    - How do you think that the Greeks would have solved the problem? (Ans: with a compass and straight edge. As would have people in Descartes' time.)
    - Why would have people in Descartes' time thought that our method was odd? (Ans: because we have turned it into an algebra problem.)
    - Descartes solved the problem the familiar way (geometry) and the new way (algebra).
  - Descartes said that with the rectangle & square problem it isn't necessary to *picture* the rectangle, and the two squares. It doesn't have to be thought as a geometry problem, but can instead be thought of as a number puzzle. However, he gives both the standard Greek geometric solution and the new algebraic solution.
    - He shows the Greek geometric solution as follows:



      - The length of the rectangle is a, and the side of the known square is b. (b must be less than half of the length of a.)
      - Draw segment NL equal to ½a. Draw a semicircle having radius NL. Draw LM equal to b and at a right angle to NL. Draw a line parallel to NL passing through M and through the semicircle in two places.
      - The possible solutions for the height of the desired rectangle are given by MQ and MR.
    - He then introduces the modern algebraic method and gets a solution of $z = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b^2}$
    - **Challenge problem**: prove that the algebraic and geometric method are equivalent.

      Proof: We begin by picturing a right triangle connecting the points N, Q, and X (midpoint of RQ).

      NQ = NL = ½a, and NX = LM = b.

      Using the Pythagorean Theorem, we see that:

      $XQ^2 = NQ^2 - NX^2 \quad \rightarrow \quad XQ^2 = (\frac{1}{2}a)^2 - b^2 \quad \rightarrow \quad XQ = \sqrt{\frac{1}{4}a^2 - b^2}$ (= XR)

      $MQ = MX - XQ \quad \rightarrow \quad MQ = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2}$

      $MR = MX + XR \quad \rightarrow \quad MR = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2}$ Q.E.D.

Descartes' Bio – Part I, up to his "day of doubting" at age 23, when he asks: "What do I know for certain?"

- Precepts and Maxims.
  - Point out the difference between what Descartes calls "precepts" (principles for action – e.g., get plenty of sleep), and "maxims" (moral rule of conduct – e.g., respect your elders).
  - <u>In Groups</u>: Discuss what your own precepts and maxims might be.

Day #4

- Review geometric (Greek) solution to square-rectangle problem.
    - If necessary, briefly show (or have students practice) using this method by solving the same problem, but with the length of the rectangle as 15 cm and the area of the known square as 36 cm^2. Redo the problem using the algebraic method. Make drawings that verify that the answers work.
    - Review that we turn geometry into algebra, whereas the Greeks thought of much math in terms of geometry.
- Review part I of Descartes' Bio. Emphasize that here we see Descartes' view on our big question:
    - How do you know anything for certain? Ans: "There are no paths to the certain knowledge of truth open to man except <u>evident intuition and necessary deduction</u>." (Descartes considered intuition "the conception of the pure and attentive mind"). But where does the intuition come from? God.
- Descartes' Bio – Part II (starting with his "day of doubting" at age 23).
    - Include that philosophy's foundation had just been shaken:
        - Planets had been shown to go around the sun.
        - The church had lost its status as the unquestionable authority.
        - Galileo had shown that Aristotle was wrong, but was put on trial in 1633.
    - Descartes sought to create a system by which he could attain "absolute certainty".
- The full title of his book (published in 1637) is *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (A Discourse on the Method of Rightly Conducting one's Reason and Searching for the Truth in the Sciences).
    - Publishing a book with such a title was a very bold thing to do. And it worked!
    - Descartes' work was a big step in the evolution of thinking.
- Hand out summary of the Discourse main text. Read Part I (very briefly), and then read Part II in more detail, which contains his <u>Four Precepts</u>:
    (1) Don't believe it until you "prove" it to yourself (i.e., think for yourself);
    (2) Solve complicated problems by the method of "divide and conquer";
    (3) Proceed from simplest to most complicated;
    (4) Don't leave anything out (i.e., collect thorough data)
- <u>In groups (and then sharing with the full class)</u> answer these questions:
    - How were views and practices of science and religion different in Descartes' time compared to now?
        - Perhaps a conclusion is that the role/position of religion and science have switched. Perhaps they say that science has become our religion (as a basis for our world view).
- The role of religion in science. In 1600, there wasn't a debate about how the universe or life was created. It *was* created by God. Kepler (In 1609) did not see the need to explain why the planets were kept in their orbits ("they were pushed by angles", meaning God kept them in their orbits). In 1805, Napoleon asks LaPlace (regarding his book on celestial mechanics) why he hasn't mentioned "the author of the universe", and LaPlace responds that he "has no need for that hypothesis".
- Descartes wants us to think for ourselves. He was opposed to the "dogmatic" ideas (e.g., through teaching) of his day. What is dogma? (One possible answer: Dogma is when ideas are fed to us and we aren't supposed to question those ideas.)
- <u>Hanging question</u>: Where do we find dogma today?

Day #5

- More on Descartes' book:
  - The main text of the book (≈60 pages) contains Descartes main thesis. It is considered to be a work of major importance in the field of general philosophy, and states his ideas on how a new approach to science ought to be practiced. This was the seed of what might be called today, the Cartesian/Newtonian scientific method.
  - The book came with three appendices: La dioptrique (Optics), Les météores (Meteorology), and La géo*métrie*. It is this third appendix for which Descartes is most famous, as he lays down the foundation of Cartesian geometry.
  - *La géométrie*, the famous third appendix, is about 100 pages long, and is divided into three sections. The first section (which is what this main lesson block concentrates on) contains an explanation of his new geometry, and shows why it is a significant advance over the Greeks. This first section is 18 pages long (up to page 37, but the pages alternate between French and the English translation), and is, itself, divided <u>into</u> 9 subsections; it deals with the rectangle & square problem and the Pappus problem. The second section deals with, among other things, a classification of curves and a method of constructing tangents to curves. The third section deals with the solutions of equations of degree greater than two. It is in this third section that he discusses what we now call *Descartes Rules of Signs*.
- Review *Discourse* main text, Part II regarding the "four precepts.":
  - Which of the four precepts do we practice today? Which don't we practice?
  - The fact that we think these are "obvious" shows how successful Descartes was. Previously, physics was based upon philosophy. Galileo introduced the idea of basing science upon experimentation and empirical data. Descartes is saying that our approach to science should be based upon mathematics; mathematics was to become a principle tool for doing science. This was a very radical idea.
- <u>What was physics before Galileo and Descartes</u>? It was largely philosophy. People would philosophize about how the world was put together – it was based upon opinion and impressions. Galileo and Descartes wanted to end this. Galileo said: "The book of nature is written in the language of mathematics."
- **<u>Descartes wanted to reduce all of science into mathematics.</u>**
- Read through Part III of *Discourse* main text, and then discuss in groups.
  Descartes' <u>Three Maxims</u>:
  1. <u>Conservatism</u>. Follow law and custom; adhere to moderate opinions; make no vows or lasting commitments.
  2. <u>Decisiveness</u>. Be firm and resolute in your actions.
  3. <u>Stoicism</u>. Instead or trying to change the world, try to change your attitude and control your desires. (Stoicism is a school of Hellenistic philosophy founded in Athens by Zeno of Citium in the early 3rd century BC. The Stoics held that emotions like fear or envy (or impassioned sexual attachments, or passionate love of anything whatsoever) either were, or arose from, false judgements and that the sage – a person who had attained moral and intellectual perfection – would not undergo them.)
- Yesterday's hanging question: Where do we find dogma today?
- Handout and discuss the <u>newspaper article on evolution</u>: "Blow to evolution as creationism tightens its grip on US schools" (by Robin McKie, Manchester Guardian science editor, Sunday February 24, 2002). What does the author think about the creationist view? How does this reflect society attitudes?
  - A sign of dogma is when people are outraged when a different view is presented.
  - This should lead to the question: can science be dogmatic?
  - Other questions for discussion: What role does science play in our society?
    What do people want from science? What isn't science able to answer?

Day #6

- Review evolution article.
- We have to be careful not to become "anti-science". There are many great scientists today. Science has made amazing progress. There are scientists today that are struggling to find the answers to the "big" questions.
- But, as our "religion", science has become dogmatic. Note an important difference between "real science" (the science being practiced by "real" scientists) and "pop science" (e.g. science as shown in the media). As it appears in "pop science" we can forget that science is a collection of theories – models of reality. People turn scientific theory into fact, and then it becomes dogma because it is unthinkable to question it. Have we (the public that is informed by pop science) forgotten the philosophical roots of science? Have we simply brushed aside the "big" questions, such as: What is life? What is consciousness? What is electricity? What is light?
- Mention that a few years ago, the title on the cover of a popular magazine read, "How the Universe Began, and How it might End." In many schools today, very young children are taught the *Big Bang Theory* as if it is a fact. Why is that? Is this dogma?
- Read through Part IV of *Discourse* main text, and then discuss in groups. Questions:
  (1) What is Descartes's answer to the question: "If I doubt everything, what can I know for certain?"?
  (2) How does he "prove" the existence of God?
  (3) How does Descartes know that our dreams are not real?

Day #7

- Hand out Book I of *La Géométrie*. Book I is divided into parts A through I. This is summarized for the teacher as follows:
  - Part A is a general introduction. He states from the start that he has a new method for solving geometric problems whereby the solutions are simply the length of straight lines.
  - Part B gives examples of simple constructions of multiplying, dividing, and taking square roots. It is three paragraphs long beginning with "For example, let AB…" at the top of page 5.
  - Part C introduces the reader to his algebraic terminology. It is three paragraphs long, starting with "Often it is not necessary…" on page 5. Descartes assumes that the reader has never seen algebraic notation before. So he needs to explain what a "+" means, what exponents are, etc.
  - Part D introduces the idea of his new method. It starts on page 6 with, "If, then, we wish …" and runs to page 13. It isn't necessary that all the details are understood.
  - Part E gives examples using the three cases of the rectangle & square problem. It starts on page 13 with "For example, if I have…".
  - Part F introduces the Pappus problem. It starts on page 17 with "This is also evident…"
  - Part G talks about how his method can solve the Pappus problem. It starts on page 25 with "First, I discovered…".
  - Part H gives a specific example on how to solve the Pappus. The is the hardest and most important section. It starts with "Now, since all the angles…" on page 29. (The end of this section contains the "seed statement" of Cartesian geometry.)
  - Part I talks briefly about his method could be used to solve more complicated versions of the Pappus problem. It starts with "This method can be used…" on page 34.
- Go through the preface of *La Géométrie* and then read through <u>sections A and B</u>. What is radical is that Descartes says that a•b can be a line. Mention that tomorrow we will practice do geometric constructions for multiplication, division, and square roots, as shown in section B in order to help us better understand this.

Day #8

- Review *La Géométrie* section B, and then, in order to understand it better, give *Cartesian Geometry, Worksheet #1*, which practices doing a few problems similar to part B.
- Hanging question: Why does Descartes' method for geometric multiplication work?
- (Otherwise, this is catch-up day!)

Day #9

Prove why the construction for a·b works.

- Go over the 4 variations (and their solutions) of quadratic equations. This is the same order as Descartes:
  - $x^2 = 13x + 30$ (This is Descartes' first example in section E: $z^2 = az + b^2$)
  - $x^2 = -13x + 30$ (This is Descartes' second example in section E: $y^2 = -ay + b^2$)
  - $x^2 = 13x - 30$ (This is Descartes' third example in section E: $z^2 = az - b^2$)
  - $x^2 = -13x - 30$ (Descartes does not give an example of this because it only gives negative results.)
- Briefly, go over part C and D. (They are not so crucial.)
  - Have the whole class quickly read *La Géométrie* section C – it will likely be incomprehensible. Then ask why it is so hard to understand. The reason: because his audience is unfamiliar with algebraic symbols and notation. So he has to explain what A+B means, what an exponent is (he is the first to write an exponent as we do it. Pay attention to how the French side has different notation than what we currently use (e.g., the "="). Now have the class quickly reread it – it should be easier to understand.
  - Simply mention *La Géométrie* section D is him saying how he can use his new geometry to solve all kinds of crazy equations (written with this new algebraic notation).
    - Read footnote #18 on page 10.
- Have students read through *La Géométrie* section E in detail, then discuss in groups. Questions to consider:
  - How many examples of quadratic equations does he cover? (Ans: 3)
  - How many drawings are there in this section? (Ans: 2)
  - What is strange about his answers? (Ans: No negative solutions)
  - Where does he get his algebraic answers? (Ans: Probably straight from the drawing. I'm not clear whether the quadratic formula – which is said to have been invented well before Descartes – would have been known and used by Descartes)
- Emphasize that Descartes didn't consider negative numbers, and therefore only showed three cases of the equation $z^2 = \pm az \pm b^2$. Also, his first two cases only had one solution, as opposed to the two solutions that modern algebra would yield.

Day #10

- Review section E.
  - Note that his third example ($z^2 = az - b^2$) is the same square-rectangle problem from last week.
  - Show how his first example ($z^2 = az + b^2$) can be seen as a variation of the square-rect problem.
  - Perhaps then convert it into an algebra problem by measuring the line segments and then plugging it into the quadratic formula. The answer for both methods should be the same.
- Do *Cartesian Geometry Worksheet #2.*
- The basic ideas of Descartes' philosophy (some of which he writes about in works other than *Discourse*).
  - Descartes' answer to the question "How do you know anything for certain?": "There are no paths to the certain knowledge of truth open to man except evident intuition and necessary deduction."
  - Descartes' new scientific method changes how we do science. (Review his 4 precepts)
  - Descartes' other key books: Discourse (1637), Meditations (1641), Principles of Philosophy (1644), Passions of the Soul (1649)

- Descartes' duality:
    1. Thinking or spiritual substance (mind).
    2. Physical substance, devoid of spirit (body).
    - Only in the human being did these two things come together. Descartes believed that in the human, mind and body are mysteriously and inextricably united, but he wasn't sure how.
- The world as a machine. "I do not recognize any difference between [man-made] artifacts and natural bodies except that the operations of artifacts are for the most part performed by mechanisms that are large enough to be easily perceivable by the senses- as indeed they must be if they are capable of being manufactured by human beings. The effects produced in nature, by contrast, always depend on structures that are so minute that they completely elude our sense….It is no less natural for a clock constructed with this or that set of wheels to tell the time than it is for a tree that grew from this or that seed to produce the appropriate fruit." From *Principles of Philosophy*

- <u>Hanging Question</u>: How do you think Descartes' idea of "World as a machine" changed things?

Day #11

<u>What is your World View</u>?

- Are you a scientist? Does science define all aspects of what you believe?
- Are you a materialist? Do you believe that only material stuff exists and that there is a physical scientific explanation for everything?
- Do you believe that a non-physical or "spiritual world" exists?
- Do you believe that non-physical beings exist? Do these beings help to direct the world and give it purpose and meaning?
- Do you believe in God?
- Do you believe that the world has purpose and meaning, perhaps guided by God or a spiritual world?
- Do you think that you have all of the answers to the above questions? Are you open to other viewpoints and open to the possibility that over time your viewpoint might change?

<u>Go over hanging question</u>: How do you think Descartes' idea of "World as a machine" changed things?

- Note how Descartes stands between the typical faithful Christian of his time and the modern scientist of today. For example:
    - Descartes transferred infallibility from the Bible and church to human reason and logic.
    - The physical world is essentially a machine, created by God, and defined by God's mechanical laws. Note: he doesn't mean the world is one huge machine, but rather that much of the physical world consists of machines – e.g., plants, etc.
    - But today Descartes' dualism has changed: the "spiritual essence" half has been largely discarded. All that is left is material, physical stuff. This is materialism taken to an extreme. Even thinking is viewed as a material/chemical process. Give the example of artificial intelligence. But perhaps now the pendulum is turning.
- Tell about how when you first look through *La Géométrie* you expect to see an intro to the Cartesian axis (x and y axes) at the beginning, but, surprisingly, it appears nowhere in the book. Our goal is to come to an understanding of how he did it.
- Go briefly through *La Géométrie* sections F and G.
- Intro to the Pappus problem – Descartes heard about it through a friend in 1631.
    - In groups work on the Pappus problem – problem #1 from *Cartesian Geometry Worksheet #3.* Some clues:
        - As soon as the 4 lines have been laid down on the page, the curve has been determined. It is your goal to make the curve visible. Imagine you are an archeologist and you are brushing away the dust to expose the hidden curve.
        - At some point, ask the students: "Can the resulting curve cross any of the lines?" (Ans: The curve can never cross one of the given lines except at one of intersection points of the four given lines; the curve passes through the intersection points of the lines WY, WZ, XY, XZ, but not through WX or YZ.)
- <u>Hanging Question</u>: leave the Pappus unfinished. Ask students to work on it for homework.

Day #12

- Review: Pappus problem – emphasize how Descartes looked at it differently than the Greeks. The Greeks would have seen it as the area of two rectangles (did any of us do that?).
  - Put the four lines of the Pappus problem on the board and then have one student at a time come to the board and locate (only) one point of the resulting curve. After a while the points can be connected to form the resulting curve.
  - Finish *Cartesian Geometry Worksheet #3*.
  - Hanging question: How can the four lines be placed on the page such that a parabola is produced? (Hint: an parabola can be considered as an ellipse where one of its ends have been stretch to infinity.)
- Introduce (or review from 8th grade) loci and conic sections.
  - What is the locus of points that are equidistant from two points? (Ans: perpendicular bisector)
  - What is the locus of points that are equidistant from a line and a point? (Ans: parabola)
  - What is the locus of points that are equidistant from a circle and a point inside that circle? (Ans: ellipse)
  - What is the locus of points that are equidistant from a circle and a point outside that circle? (Ans: hyperbola) Note: this only produces one branch of the hyperbola.
- Have students read Fritjof Capra's essay (*The Turning Point*, pp56-62) on Descartes' influence on science.
- Discuss how the analytical/reductionist method has influenced many branches of science, including medicine, economics, biology, politics, etc. Point out the many amazing accomplishments of humanity (e.g. putting a man on the moon) that this approach made possible, as well as the limitations and drawbacks. (See also, Fritjof Capra's book, *The Turning Point*.) Include how it changed our view of the sun and the heart, as well as the human body.

Day #13

Give summary of section H (see below).

  - Show in detail, to the whole class, how he calculated CD.
  - Those students wanting a challenge should work through the whole sheet and write an entry in the their main lesson books on it.
- Although most of Descartes's geometry in his book is completely unfamiliar to us today, there are two sentences in his book that are the seed for Cartesian geometry:
  - p29 (end of section G): "Call the segment AB, x, and call BC, y." This means (looking at his drawing) that in order to find a desired point on the curve, A is the origin. Then we go out a distance of x along the horizontal line, and then a distance y (which we have determined after given a value for x) along the line BC.
  - P34: "We may give any value we please to either x or y, and then find the value of the other from this equation."
- Emphasize the sentences (on the top of p34) that form the seed of modern Cartesian geometry.
- Key Idea: Descartes' method for his new geometry (i.e., plotting a curve) is about as impractical as it gets, but it led to one of the most practical creations in all of mathematics.
- Yesterday's hanging question: Show the drawing (from appendix A in this source book) titled "The Pappus Loci Problem in Movement", which shows the transformation of an ellipse, to a parabola, to a hyperbola by rotating line x. (It turns out that the parabola is produced when line x is at an angle of about 77⅓°.)
- Show my drawing of the solution of the Pappus problem. End by giving the variation where three lines are vertical (and line x is the leftmost), and the fourth line, w, is horizontal.

Summary of Cartesian Science (from Capra p53-62):

- Much of the world is still stuck in the thinking of 100 years ago. The new thinking brought to us by modern physics has not yet penetrated other realms.
- Descartes' ideas were the first step from seeing the world as an organism to seeing it as a machine.
- These were radically new thoughts:
  - The sun was just a large ball of burning gas, devoid of any spiritual essence.
  - The heart was just a pump.
- Newton started where Descartes left off. He was critical of Descartes, and further refined the new scientific approach. After Newton, science reigned supreme, above philosophy and theology. If you wanted to really understand the world, you needed to understand science.
- This new scientific method, from the work of Newton and Descartes, is:
  - Experiment based (Galileo)
  - Empirical (Descartes)
  - Objective (Newton)
- Give review sheet.
- Hanging Question: How did Descartes' geometry transform the worlds of mathematics and science?

Give an overview History of Science (mostly biology):

- 1543: Nicolaus Copernicus, heliocentric theory published\
- 1604: Galileo Galilei, distance for falling object increases as square of time
- 1628: William Harvey's book *On the Movement of the Heart* begins the idea that the heart is a pump.
- 1637: Descartes' book *Discourse on the Method* begins the idea that the world is a machine.
- 1680: Giovanni Borelli (student of Galileo) writes a book describing muscles in mechanistic terms.
- 1687: Isaac Newton, publishes laws of motion and gravitation
- 1745: The Leyden jar (static electricity)
- 1769: James Watt's steam engine
- 1770's: Oxygen is discovered.
- 1800: Volta's pile (battery)
- 1819: Ørsted's Experiment (electromagnetism)
- 1821: Michael Faraday, first electric motor
- 1859: Charles Darwin publishes *On the Origin of the Species by Means of Natural Selection.*
- 1871: Darwin publishes *The Descent of Man*
- 1878: Louis Pasteur publishes *Germ Theory and Its Applications to Medicine and Surgery*.
- 1881: First vaccine.
- 1885: First successful appendectomy is performed.
- 1915: Albert Einstein, general relativity
- 1928: Penicillin is discovered as the first antibiotic.
- 1945: atomic bomb
- 1954: First organ (kidney) transplant.
- 1978: First successful "In vitro fertilisation."
- 1996: The first cloning of a mammal, a sheep named Dolly.

Day #14

- Review: especially the sentence that forms the seed of modern Cartesian geometry.
- Give the *Golden Rules of Cartesian Geometry*: (1) For every solution to the equation there is a point on the graph, and (2) Every point on the graph is a solution to the equation, and vice-versa.
- Go over section I (briefly).
- Note that Descartes' method derives complexity from the number of lines and their positions, whereas modern Cartesian geometry gets complexity from the equation; the axes are always the same.
- Yesterday's hanging question: Talk about how Cartesian geometry changed things (also, see *Descartes' new geometry*, below):
  - Geometry and Algebra were now united.
  - Geometry could now be reduced to Algebra.
  - Any geometrical figure could now be expressed simply as an equation.
  - Many other things could be expressed as an equation, such as population growth, the beating of the heart, aspects of weather, the economy, etc.
- Go over review sheet.

Descartes' new geometry:

- *What did Descartes do?* Through the Pappus problem, he found a method by which he could choose an x value and then determine how far out along the y line to go in order to locate the point on the curve that made the equation work.
- *How did he do this?* He came up with a general (and very complex) equation that related x and y.
- *Why was this significant?* It was the first time that anyone had attached as equation to a curve, and *it united algebra and geometry.*
- Many of Descartes' contemporaries missed the point of *La Géométrie* and thought it was primarily concerned with geometric constructions. Leibniz said it was "a regression to the Greeks".
- You can choose any value for x and then determine a corresponding value for y, and then plot that point.
- In theory, I can do this with any equation that is expressed in two variables.
- *The Golden Rule of Cartesian Geometry*: Every solution to the equation is a point on the graph, and v.v.
- Cartesian geometry allows us to:
  - *Visualize* the seemingly random infinite solutions of an equation expressed in two variables. For example, $4x^2 - 9y^2 + 24x + 36y + 36 = 0$ has solutions, that when plotted, form a curve called a hyperbola.
  - Assign an equation to any geometric shape. *Geometry is reduced to algebra!* In a similar manner, the behavior of practically anything (the beating of the heart, the weather, aspects of the economy, etc.) can be expressed in (reduced to) an equation.
  - Reduce many other things to an equation: such as population growth, the beating of the heart, aspects of weather, the economy, etc.
- *The Fork in the Road*

  For nearly 2000 years, there had been no significant developments in the field of geometry since Euclid. Now, suddenly, at the beginning of the 17th century, two men were offering humanity a new path – two completely new ways of thinking. Descartes was offering analytical geometry, a powerful new method that could reduce geometric form into algebraic formulas. Desargues (with the support of Pascal) was offering the beginnings of non-Euclidean geometry (projective geometry) – a geometry that was detached from measurement and the physical world, and largely lived in the realm of imagination. These two geometries were polar opposites of one another. Humanity choose to go down the path of Cartesian geometry; non-Euclidean geometry was not to be taken up again for almost 250 years, long after Cartesian geometry had made a huge impact on the development of humanity.
- <u>Key Idea</u>: Descartes' method for his new geometry (i.e., plotting a curve) is about as impractical as it gets, but it led to one of the most practical creations in all of mathematics.

Extra Days??

- Spend a day or two giving an intro to modern Cartesian geometry, and do worksheet #4. Otherwise, an intro to modern Cartesian geometry can be done in track class.

Descartes' Biography:

- The biography demonstrates that one's life can have a central purpose to fulfill for humanity. Ask the students: What do you think this life-changing event was for Descartes? The central event in Descartes' life is his dream/revelation that occurred when he was 23. Before 23, everything led him to that point. Afterwards, everything consisted of carrying out that mission.
- Born (March 31, 1596) the second son of an aristocratic, but not very wealthy, family in La Hayes (now Descartes) in central France. Descartes himself remained Catholic his whole life.
- Mother died (when René was not yet 2) after childbirth from the next child (who also dies a year later). Father remarries and moves away. René and his brother and sister are then raised by a nurse at grandmother's and then great-uncle's house. Doesn't stay at one place very long. Generally, he is unhealthy.
- Goes to boarding school (at 10 years old) at La Flèche, 100 km away (two days travel?). It was a Jesuit boarding school with about 1200 students. Above all, the school teaches discipline and how to be a good Catholic. It is considered one of the best educations that you could have. (This is at a time of great struggle with the Protestants.) It is said that he develops his habit of lying in bed thinking because the head of the school allowed René to sleep in because of his poor health.
- Descartes' childhood was under the reign of Henri IV, who was a Huguenot, but then converted to Catholicism just before his coronation in 1589. In 1598, Henri "enacted the Edict of Nantes, which guaranteed religious liberties to the Protestants and thereby effectively ended the civil war… Henry showed great care for the welfare of his subjects and displayed an unusual religious tolerance for the time. He was assassinated by a fanatical Catholic" in 1610 when Descartes was 14.
- Finishes his law degree at age 20. However, second sons either joined the church or the army. He chose to join the Dutch army (Maurice of Nassau, Prince of Orange) at the age of 22. It was a safe bet because the Netherlands had recently signed a 12 year truce with Spain.
- While in Maurice's army, he was stationed outside Breda (today in the south of the Netherlands).
  On Nov 10, 1618, he meets Isaac Beeckman (philosopher and scientist) on the streets of Breda while looking at a poster of a challenge to solve a math problem. Their (stormy) friendship is considered the beginning of Descartes' intellectual career.
- He then joins the army of the Duke of Bavaria. He is stationed near Ulm (southern Germany) to take part in the ceremony for the coronation of Ferdinand II as the Holy Roman Emperor (which was on Sept. 9, 1619).
- He then rents a "stove-heated room" to wait out the winter. On Nov 10, 1619, he lies in bed the whole day. He reviews his knowledge, and this leads him to ask: "what do I know for certain?" He realizes that he has "progressed no further than to recognize my own ignorance". He then doubts everything, even the existence of his own body. Then comes the Cogito, etc. That night he has three dreams, which reinforce he belief that God had given him this mission.
  (1) He was battling a strong wind which was blowing him away from a church.
  (2) A storm which ended with a great clap of thunder which filled his room with sparks.
  (3) He sees two books: one is a large dictionary, the other is a book of poems, from which he reads a poem beginning with "What path shall I follow?"
  He believes these dreams are a revelation from God where he has perceived the foundations of a new science (which he develops over the next 20 years).
- He then spent the 10 years (from age 22 to 32) in various armies and traveling through 10 countries and occasionally visiting France. He is torn between having a good time with his friends and spending time alone contemplating things. During this time he meets Father Marin Mersenne, Pascal, and Desargues.
- Spends the next 20 years (from age 32 to 52) in Holland. During that time he his very reclusive, but still moves about once per year. It is during this time that he does his writing (Discourse, age 41; Meditations, age 45; Principles of Philosophy, age 48) He is in regular contact with Mersenne, and has a correspondence with Princess Elisabeth of Bohemia during the last 7 years of his life. He and Helena Jans (a servant) live together starting in 1634, and soon after have a child, Francine. Francine died at age 5 of a fever.
- In 1643, Cartesian philosophy was condemned at the University of Utrecht. In 1663, the Pope placed his works on the *Index of Prohibited Books*.
- He returns to Paris briefly in 1648 to receive a pension from the king, but finds political upheaval, and then goes back to Holland. In February 1649, Descartes is invited to Sweden by Queen Christina (age 22) (the "Girl King", who had ruled since age 6). He finally goes in August, but the weather and having to get up at 5am to give the Queen lessons, kill him. He died Feb 11, 1650 – one year after arriving in Sweden.
- Today, he is considered the father of analytical geometry and of modern philosophy, and he redefined how we think of and practice science. After Descartes died in Sweden, Queen Christina abdicated her throne to convert to Roman Catholicism (Swedish law requires a Protestant ruler).

Section H of *La Géométrie*

The Key Ideas

- This is a summary of what Descartes did (as described in *La Géométrie*, section H) in his treatment of the Pappus problem. The whole section tells in detail how you can find one point, C, on the resulting curve.
- We begin with the idea that from point A (the origin) we can travel any given distance, x, along the line EG, and from there (point B) travel a certain distance, y, at the required angle (θ_3) until we intersect (at point C) the desired loci curve.
- It is important to note that if we imagine point C moving along the desired loci curve, then all of the triangles (see below) will retain their same shape, angles, and ratios, even though their size will change.
- The goal is to derive a formula that, given any x value, calculates y, or, given any value for y, calculates x.
- The four lines are placed on the page, and we are to find point C on the curve such that the distances to each line – measured at the required angles (θ_1, θ_2, θ_3, θ_4) – satisfy the equation **CB·CF = CH·CD**.

Triangle Ratios (Descartes has chosen to express all of his ratios in terms of z, which could be any number.)

Using ΔARB, let AB:BR = z:b.

$BR = \frac{b \cdot AB}{z}$ since AB = x we get $BR = \frac{b \cdot x}{z}$

Using ΔDRC, let CR:CD = z:c. $CD = \frac{c \cdot CR}{z}$

Using ΔESB, let EB:BS = z:d.

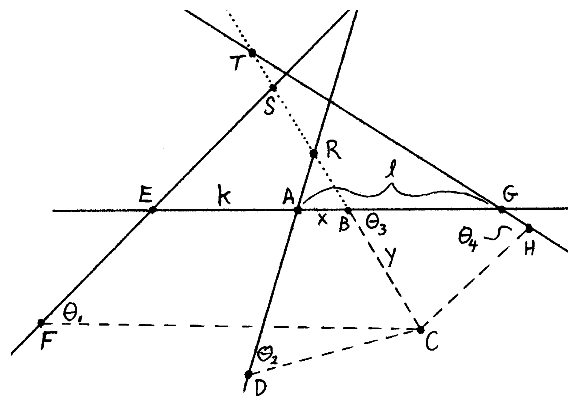
$BS = \frac{d \cdot EB}{z}$ since EB = k+x we get $BS = \frac{dk + dx}{z}$

Using ΔFSC, let CS:CF = z:e. $CF = \frac{e \cdot CS}{z}$

Using ΔBGT, let BG:BT = z:f.

$BT = \frac{f \cdot BG}{z}$ since BG = $\ell - x$ we get $BT = \frac{f\ell - fx}{z}$

Using ΔTCH, let TC:CH = z:g. $CH = \frac{g \cdot CT}{z}$



Deriving Formulas for the Four Distances (CB,CF, CH, CD)

- Calculating CB. $\boxed{CB = y}$ (The rest aren't that easy!)
- Calculating CD.
  CR = BR+BC. Given BC=y and $BR = \frac{b \cdot x}{z}$ (from above), we get $CR = \frac{b \cdot x}{z} + y$

  $CD = \frac{c \cdot CR}{z}$ (from above). Substituting, we get $CD = \frac{c \cdot \left(\frac{b \cdot x}{z} + y\right)}{z}$ → $CD = \frac{cy}{z} + \frac{bcx}{z^2}$ (the second c is missing in the translation) which becomes $\boxed{CD = \frac{czy + bcx}{z^2}}$
- Calculating CF.
  CS = BS + CB. Given CB = y and $BS = \frac{dk + dx}{z}$ (from above), we get $CS = \frac{dk + dx}{z} + y$ → $CS = \frac{dk + dx + zy}{z}$

  $CF = \frac{e \cdot CS}{z}$ (from above). Substituting, we get $CF = \frac{e \cdot \left(\frac{dk + dx + zy}{z}\right)}{z}$ which becomes $\boxed{CF = \frac{ezy + dek + dex}{z^2}}$
- Calculating CH.
  CT = BT + CB. Given CB = y and $BT = \frac{f\ell - fx}{z}$ (from above), we get $CT = \frac{f\ell - fx}{z} + y$ → $CT = \frac{zy + f\ell - fx}{z}$

  $CH = \frac{g \cdot CT}{z}$ (from above). Substituting, we get $CH = \frac{g \cdot \left(\frac{zy + f\ell - fx}{z}\right)}{z}$ which becomes $\boxed{CH = \frac{gzy + fg\ell - fgx}{z^2}}$

Section H of *La Géométrie* (continued)

- The first sentence at the top of page 34 restates the condition of the Pappus problem: CB·CF = CH·CD. This can therefore be expressed as:

$$y \cdot \left(\frac{ezy + dek + dex}{z^2}\right) = \left(\frac{gzy + fg\ell - fgx}{z^2}\right) \cdot \left(\frac{czy + bcx}{z^2}\right)$$

- In the first sentence on page 33, he says that with each of the above formulas x and y are the only variables, and all of the rest (c, b, z, e, d, e, g, f) are given, or easily calculable. Then he says (p34) that **we can assign a value to y, and because all the other variables except x are known, this equation can then be reduced to the form of $x^2 = \pm jx \pm k^2$**, which is something that he has shown us how to solve earlier. We have therefore achieved our goal of deriving a formula that calculates x, given any y value. (Likewise, we could randomly choose any value for x, and then calculate the corresponding value for y, which would yield a point on the curve.)

- The essence of what we know to be Cartesian geometry today is found in the two sentence on page 34:
  "We may give any value we please to either x or y and find the value of the other from this equation…If then we should take successively an infinite number of values for the line y, we should obtain an infinite number of values for the line x, and therefore an infinity of different points, such as C, by means of which the required curve could be drawn."

Suggested Reading:

- *La Géométrie*, René Descartes, Pages 2-37, Dover Publications, 1954
- *A Discourse on Method* (combined in one volume with *Meditations and Principles*), René Descartes, Pages 3-57, Everyman Library, 1994
- *The Passion of the Western Mind*, Richard Tarnas, Pages 248-322, Ballantine Books, 1991
- *The Turning Point*, Fritjof Capra, Pages 53-62, Bantam Books, 1982
- *Descartes – an Intellectual Biography*, Stephen Gaukroger, Oxford University Press, 1997
- *Cogito Ergo Sum*, Richard Watson, David R. Godine Publisher, 2002

The Evolution of Cartesian Geometry:

- ***TBC Read Kline p302, 317-324 and Boyer p346. Include Newton's and Fermat's roles.