Cantor's Set Theory

Proof that <u>the rational numbers are countable</u> (which means they can be put into a one-to-one correspondence with the counting numbers):

- Make a 2-D array which encompasses all of the rational numbers in the following way: the first row is $^1/_1$, $^2/_1$, $^3/_1$, etc. the second row is $^1/_2$, $^2/_2$, $^3/_2$, etc. the third row is $^1/_3$, $^2/_3$, $^3/_3$, etc.
- Now we can number all of them along diagonals so that #1 is $^1/_1$, and #2 is $^2/_1$, #3 is $^1/_2$, #4 is $^3/_1$, #5 is $^2/_2$, #6 is $^1/_3$, #7 is $^4/_1$ #8 is $^3/_2$, etc.
- In this way all of the rational numbers can be mapped onto the counting numbers.

Proof that <u>the algebraic numbers are countable</u>:

- The algebraic numbers are the complex number solutions to the algebraic equations in the form of: $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0x^0 = 0$, where a_i is an integer, and n is a positive integer.
- Our goal is to find a systematic method for putting all of the algebraic in some specific order, thereby matching them in a one-to-one correspondence with the counting numbers.
- Let A(i) be the i^{th} algebraic number.
- Let $K = n + |a_n| + |a_{n-1}| + |a_{n-2}| + \ldots + |a_1| + |a_0|$, where K represents the "index" of the equation.
- Usually, there are several algebraic equations for a given index, K. Some of these equations have multiple solutions. Some of these solutions will be ignored because they are algebraic numbers that were found with an equation from a smaller index. Of all of the new solutions to the equations produced by a given index, in the form a + b*i*, we will put them into a specific order by arranging the numbers according first to the size of b, and then, in the case of equal b values, according to the size of a.
- There are no equations where K = 0 or K = 1 that yield any algebraic numbers as solutions.
- Of the equations where K = 2, only one algebraic number (x = 0) is produced. Therefore A(1) = 0
- Of the equations where K = 3, only two new algebraic numbers (x = –1, 1) are produced. Therefore A(2) = –1, and A(3) = 1
- Of the equations where K = 4, six new algebraic numbers (x = ±2, ±½, ±*i*) are produced. Therefore A(4) = –2, A(5) = –½, A(6) = ½, A(7) = 2, A(8) = –*i*, A(9) = *i*.
- Of the equations where K = 5, twenty new algebraic numbers (including $x = \pm\sqrt{2}, \pm\sqrt{2}\,i, \pm\phi$) are produced, thereby giving us the values of A(9) through A(28).
- This can be continued indefinitely for all values of K.
- Therefore all of the algebraic numbers can be put into a one-to-one correspondence with the counting numbers.

Proof that <u>the real numbers are not countable</u>:

- Every real number has a specific and unique decimal representation.
- <u>Assumption</u>: The real numbers between 0 and 1 are countable.
- It is therefore possible to write down <u>all</u> of the real numbers between 0 and 1 in a systematic order from $a_1, a_2, a_3, \ldots$ etc., thereby matching them in a one-to-one correspondence with the counting numbers.
- Let d be a 2-D array such that for any number, a_n, the i^{th} digit of a_n is given by $d_{n,i}$. This means that the 4^{th} number in our list, a_4, has a decimal expansion of: $a_4 = 0\,.\,d_{4,1}\,d_{4,2}\,d_{4,3}\,d_{4,4}\,d_{4,5}\ldots$ (So if $a_4 = 0.463872048\ldots$, then $d_{4,6} = 2$)
- Now we will create a new number, j, between 0 and 1. We assign its first digit to be anything but $d_{1,1}$ and the 2^{nd} digit to be anything but $d_{2,2}$, and the 3^{rd} digit to be anything but $d_{3,3}$, etc.
- j is not equal to a_1 because they have different first digits; j is not equal to a_2 because they have different second digits; j is not equal to a_3 because they have different third digits, etc. j is not equal to any number on our list. We have therefore found a real number between 0 and 1 that is not on the list, so the list does not include all real numbers. Therefore, our assumption is false; the real numbers are *not* countable.
- <u>Corollary</u>: There exists at least one irrational number that is not algebraic. (These numbers are called *transcendental numbers*.)
- <u>Corollary</u>: The transcendental numbers are *not* countable. (Loosely speaking, this means that there are "far more" *transcendental numbers*, than there are rational or algebraic numbers.)
- <u>Corollary</u>: The transcendental numbers and the irrational numbers have the same order, or *cardinal number* (i.e., they can be mapped into a one-to-one correspondence with each other).