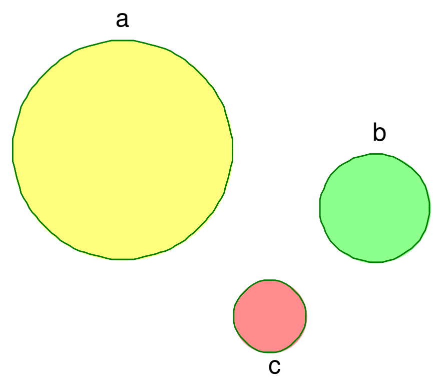
## 32. The Tangency Problem of Apollonius.

**Construct all circles tangent to three given circles.**

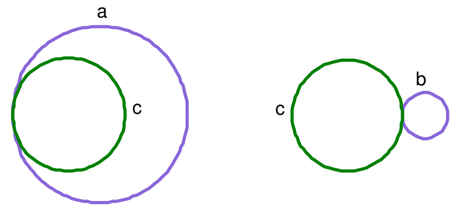



This celebrated problem was posed by Apollinius of Perga (ca. 260-170 BC), the greatest mathematician of antiquity after Euclid and Archimedes. His major work *Koniká* extended with astonishing comprehensiveness the period's slight knowledge of conic sections. His treatise *De Tactionibus*, which contained the solution of the tangency problem stated above, has unfortunately been lost. François Viète, called Vieta, the greatest French mathematician of the sixteenth century (1540-1603), attempted in about 1600 to restore the lost treatise of Apollonius and solved the tangency problem by treating each of its special cases individually, deriving each successive one from the preceding one. In contrast to this, the solutions of Gauss (*Complete Works*, vol. IV, p.399), Gergonne (*Annales de Mathématiques*, vol. IV), and Peresen (*Methoden und Theorien*) solve the general problem.

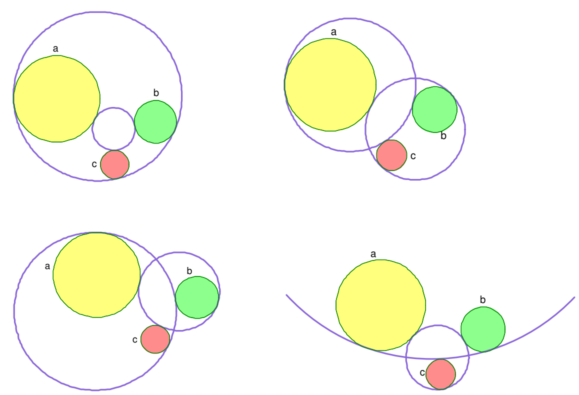
We restrict ourselves to the elegant solution of Gergonne. His solution is based on the device of seeking the unknown circles in pairs, rather than individually.

**Note 1**. The construction is complicted enough and (as written by Dörrie) involves so many likely unfamiliar concepts and terms, that I will explain just enough things so that the interested reader can make his or her constructions. It's a good exercise to do this with geometry software. Needless to say, what follows is not an exact or even approximate translation of Dörrie, but the ideas are the same.

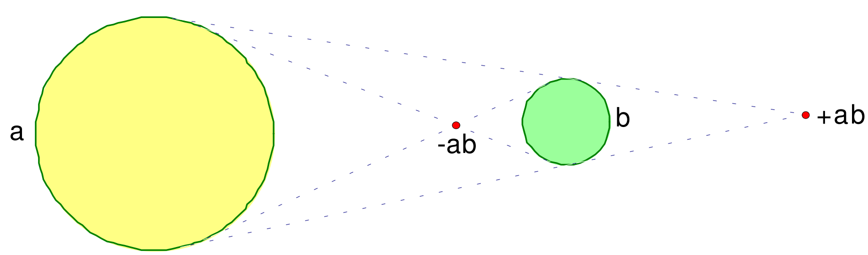
In general there are eight circles tangent to three given circles. Here's why. When there are just two circles, one can entirely surround the other or not.




Because of the nature of the problem, we know that the tangent circles cannot be inside any of the three given circles. Since a tangent circle to each of the three given circles can be one to two type, there are $2^3 = 8$ possible tangent circles.




For two circles $a$ and $b$, let $+ab$ denote the point of intersection of their two external tangents, and $-ab$ denote the point of intersection of their two internal tangents.



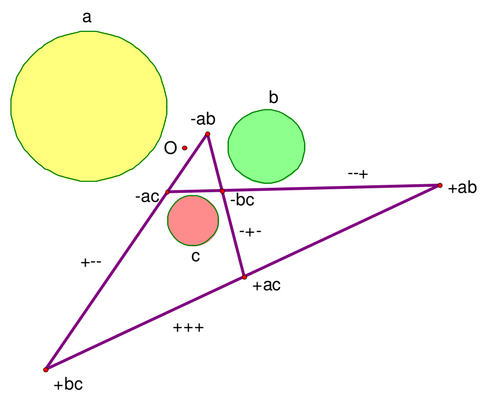

The construction of the tangent circles requires

**The Power Center** $O$ of three circles $a, b, c$. This is the intersection of the radical axes of pairs of circles. (See No. 31.)

**Monge's Theorem**. If three circles $a, b, c$ are taken in pairs $bc, ca, ab$, the points at which the external tangents to each pair are collinear; similarly the external intersection point of one pair, and the intersection points of internal tangents to the other pairs are also collinear. These lines are called the similarity axes of the three circles.

**Note 2.** Dörrie credits this theorem to d'Alembert; most books call it Monge's Theorem, perhaps for his simple proof.

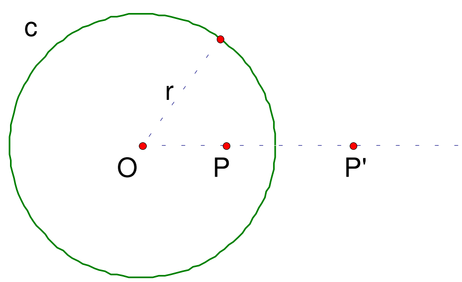
**Note 3.** If $\alpha, \beta, \gamma$ are each $\pm 1$ and $\alpha\beta\gamma = 1$, then Monge's Theorem can be stated in the form $\alpha bc, \beta ca$ and $\gamma ab$ are collinear. I will denote this line by $\alpha\beta\gamma$ with the $1$ suppressed, so for example, the line for through $-bc, +ca, -ab$ is $-+-$.




Similarity axes

We also need some results about inversion in a circle.

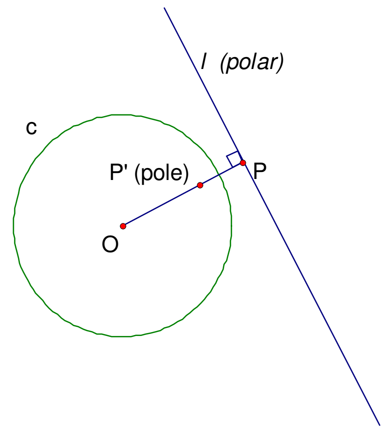
**Definition.** Let $c$ be a circle with center $O$ and radius $r$. Two points $P$ and $P'$ on a ray from $O$ are inverse points if $OP \cdot OP' = r^2$. $P'$ is the inverse of $P$ and $P$ is the inverse of $P'$.




OP.OP' = $r^2$

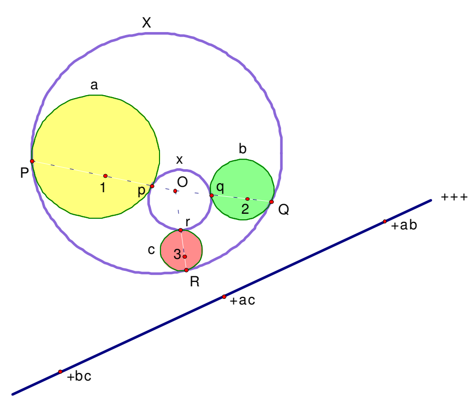
Inverse points

**Definition.** Let $l$ be a line in the plane of circle $c(O, r)$ center $O$ radius $r$. Let $OP \perp l$ with $P$ on $l$. The inverse of $P$ with respect to $c$ is the *pole* (point) of the *polar* (line $l$).

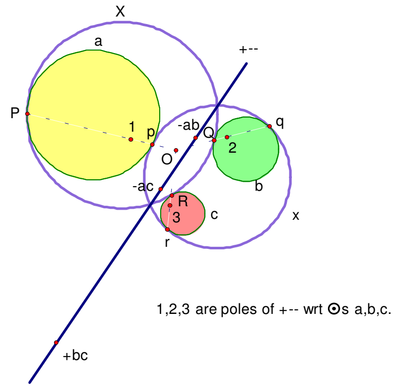



Here is **Gergonne's construction** for the circles of Apollonius:

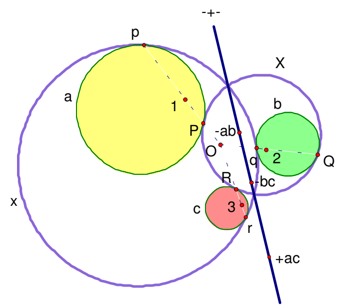
1. Construct the power center $O$ of circles $a,b,c$.
2. Construct a similarity axis $\chi$ of $a,b,c$.
3. Construct the poles $1,2,3$ of $\chi$ with respect to $a,b,c$ respectively.
4. Construct points of intersection of
   a. ray $O1$ with circle $a$, say $p$ and $P$ in the order $OpP$ if $\alpha = +1$ and the order $OPp$ if $\alpha = -1$,
   b. ray $O2$ with circle $b$, say $q$ and $Q$ in the order $OqQ$ if $\beta = +1$ and the order $OQq$ if $\beta = -1$, and
   c. ray $O3$ with circle $c$, say $r$ and $R$ in the order $OrR$ if $\gamma = +1$ and the order $ORr$ if $\gamma = -1$.
5. The circles $X$ through $P,Q,R$, and $x$ through $p,q,r$ are tangent to circles $a,b,c$.
6. Repeat steps $2$ through $5$ with another similarity axis (until all four have been used).



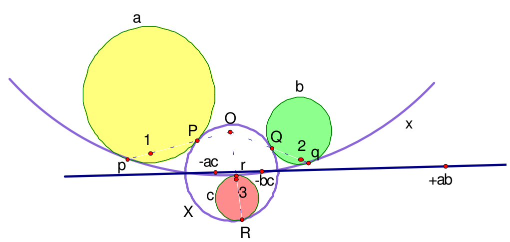

1,2,3 are poles of +++ wrt ⊙s a,b,c.




1,2,3 are poles of +-- wrt ⊙s a,b,c.




1,2,3 are poles of -+- wrt ⊙s a,b,c.




1,2,3 are poles of --+ wrt ⊙s a,b,c.

**Note 4**. The construction of course requires proof. Details can be found in Dörrie.

**Note 5**. One of more of the circles $a, b, c$ may be a point or line. See for example *Geometry: Euclid and Beyond* by R. Hartshorne, Springer, 2000.