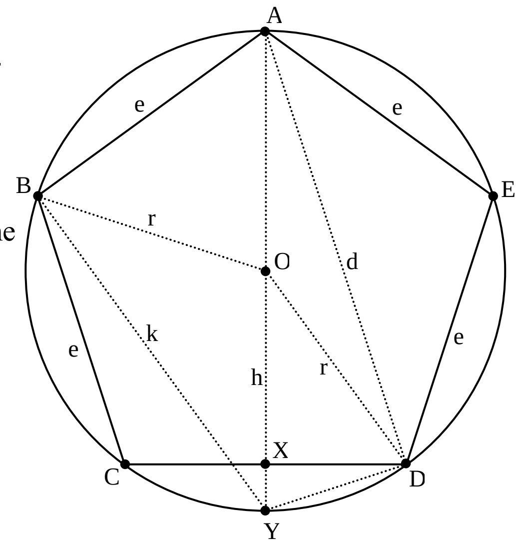
The Ratios of a Pentagon



- *Diagonal : Edge*

  $d : e = \Phi : 1$ where $\Phi = \frac{1+\sqrt{5}}{2}$

  (See the proof under "The Golden Ratio and the Pentagon" in the 10th Grade *Sequences and Series* unit.)

- *Circumradius : Short-chord* (which is YD)

  Because $\Delta ODY \sim \Delta ACD$ we can show that **$r : YD = \Phi : 1$**

- *Circumradius : Edge*

  $\angle ADY = 90° \rightarrow AY^2 = AD^2 + YD^2$

  and since $AY = 2r$; $AD = \Phi e$; $YD = r/\Phi$

  we get $(2r)^2 = (\Phi e)^2 + (r/\Phi)^2$

  which leads to $r : e = \frac{\Phi^2}{\sqrt{4\Phi^2 - 1}} : 1$

  and eventually simplifies to **$r : e = \frac{\sqrt{50+10\sqrt{5}}}{10} : 1$** and the reciprocal is **$e : r = \frac{\sqrt{10-2\sqrt{5}}}{2} : 1$**

- *Inradius : Edge* (The "Inradius", shown as h, is the radius of the inscribed circle)

  Using the right ΔOXD, we get $OD^2 = XD^2 + OX^2$

  and since $OX = h$; $XD = ½e$; $OD = r = e \cdot \frac{\sqrt{50+10\sqrt{5}}}{10}$

  we get $\left(\frac{e\sqrt{50+10\sqrt{5}}}{10}\right)^2 = (½ e)^2 + h^2$ which eventually simplifies to **$h : e = \frac{\sqrt{25+10\sqrt{5}}}{10} : 1$**

- *Circumradius : Inradius*

  Given the above two ratios, we can say $r : h = \frac{\sqrt{50+10\sqrt{5}}}{10} : \frac{\sqrt{25+10\sqrt{5}}}{10}$

  which eventually simplifies to $r : h = \sqrt{6-2\sqrt{5}} : 1$ which, in this case, can be simplified using Bhāskara's Identity[1] to **$r : h = (\sqrt{5}-1) : 1$** with a reciprocal of **$h : r = \frac{1+\sqrt{5}}{4} : 1$**

- *Mid-diagonal : Circumradius* (The "Mid-diagonal" is shown as k)

  Because $\Delta BOY \sim \Delta AED$ we can show that **$k : r = \Phi : 1$**

- *Mid-diagonal : Edge*

  Since $k = \Phi \cdot r$, we can use the above ratio we found for $r : e$, multiply it by Φ,

  and after simplifying, we get **$k : e = \frac{\sqrt{25+10\sqrt{5}}}{5} : 1$**

- *Mid-diagonal : Inradius*

  Comparing the above ratios $k{:}e$ and $h{:}e$ we can now conclude that **$k : h = 2 : 1$**

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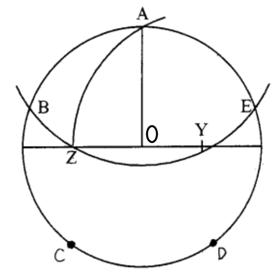
[1] Bhāskara's Identity is $\sqrt{a \pm \sqrt{b}} \equiv \sqrt{\frac{a+x}{2}} \pm \sqrt{\frac{a-x}{2}}$ where $x = \sqrt{a^2 - b}$

This allows us to simplify a compound radical of the form $\sqrt{a \pm \sqrt{b}}$ if $a^2 - b$ turns out to be a perfect square. With the problem at hand, $\sqrt{6-2\sqrt{5}}$, which is the same as $\sqrt{6-\sqrt{20}}$, $a^2 - b = 16$, so $x = 4$. Bhāskara's Identity then gives us $\sqrt{\frac{6+4}{2}} - \sqrt{\frac{6-4}{2}}$ which simplifies to $\sqrt{5} - 1$. With most other compound radicals, such as $\sqrt{25-10\sqrt{5}}$, which is the same as $\sqrt{25-\sqrt{500}}$, $a^2 - b$ is not a perfect square, so Bhāskara's Identity is not helpful.

Various Methods for **Constructing a Pentagon**

Ptolemy's Construction (ca. 150AD)



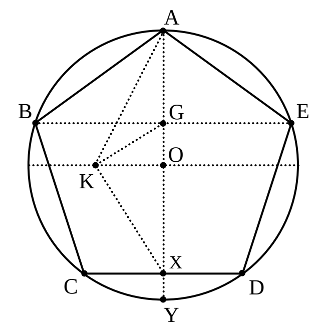
Construction: Draw a horizontal diameter and find the midpoint, Y, of the radius. A is located vertically above the center. Find Z on the diameter by drawing an arc through A with Y as the center. AZ is the desired length of the edge of the pentagon.

To Prove: *If the radius of the circle is 1, then AZ =* $\frac{\sqrt{10-2\sqrt{5}}}{2}$ *, which is the required length of the edge of the pentagon.* (See "The Ratios of a Pentagon" on the previous page.)

Proof: $AY^2 = AO^2 + OY^2 \rightarrow AY^2 = 1^2 + (½)^2 \rightarrow$ **AY =** $\mathbf{\frac{\sqrt{5}}{2}}$ AY = ZY and OZ = ZY – OY $\rightarrow$ **OZ =** $\mathbf{\frac{\sqrt{5}-1}{2}}$

$AZ^2 = AO^2 + OZ^2 \rightarrow AZ^2 = 1^2 + \left(\frac{\sqrt{5}-1}{2}\right)^2$ which leads to **AZ =** $\mathbf{\frac{\sqrt{10-2\sqrt{5}}}{2}}$ *Q.E.D.*

Richmond's Construction (by H. W. Richmond in 1893)



Construction: Draw a horizontal diameter and then find the midpoint, K, of the radius. Draw a vertical diameter to locate points Y and A (the first point of the pentagon) on the circle. Bisect ∠AKO and mark G where the bisector crosses AO. Bisect the external angle to ∠AKO (i.e., bisect the angle formed by KO and AK extended) and mark X where this bisector crosses OY. (Note that ∠GKX is a right angle.) Draw horizontal lines through points G and X in order to locate the four remaining points of the pentagon.

To Prove: *If the radius of the circle is 1, the lengths of the five edges of the pentagon ABCDE are all equal to* $½ \cdot \sqrt{10-2\sqrt{5}}$. (See "The Ratios of a Pentagon" on the previous page.)

Proof: Using ΔKAO, if we let AO = 1 then OK = ½ and AK = $\frac{\sqrt{5}}{2}$ Using the Triangle Angle-Bisector Theorem, we get AG : OG = AK : OK = $\sqrt{5}$: 1. Using Euclid V-18 $\rightarrow$ (AG + OG) : OG = ($\sqrt{5}$ + 1) : 1

which gives us AO : OG = ($\sqrt{5}$ + 1) : 1 $\rightarrow$ OG = $\frac{\sqrt{5}-1}{4}$ and AG = AO – OG $\rightarrow$ AG = $\frac{5-\sqrt{5}}{4}$

BG is a leg to two right triangles $\rightarrow BG^2 = BO^2 - OG^2 = AB^2 - AG^2 \rightarrow AB^2 = BO^2 - OG^2 + AG^2$

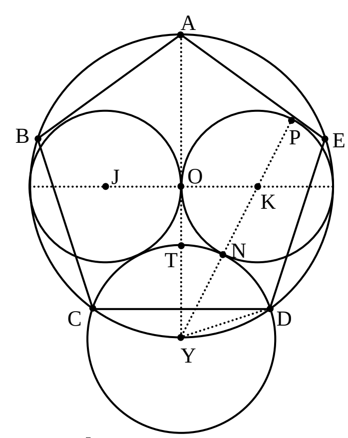
which gives us $AB^2 = 1^2 - \left(\frac{\sqrt{5}-1}{4}\right)^2 + \left(\frac{5-\sqrt{5}}{4}\right)^2$ Solving for AB gives the desired **AB =** $\mathbf{\frac{\sqrt{10-2\sqrt{5}}}{2}}$

We can now use the *Altitude of the Hypotenuse Theorem* with the right ΔGKX $\rightarrow KO^2 = OG \cdot OX$

Now ¼ = $\frac{\sqrt{5}-1}{4} \cdot OX \rightarrow OX = \frac{\sqrt{5}+1}{4}$ Using right ΔOXC leads to CX = $\frac{\sqrt{10-2\sqrt{5}}}{4}$ and **CD =** $\mathbf{\frac{\sqrt{10-2\sqrt{5}}}{2}}$

It can then be shown that the rest of the edges of the pentagon also have the same length. *Q.E.D.*

Hirano's Construction (by Yosifusa Hirano in the 19th century)



Construction: Along the horizontal diameter of the given circle, draw two half-sized circles (with centers J and K). Draw line YK intersecting the half-sized circle at points N and P. Using Y as the center, and YN as the radius, draw a circle, which locates points C and D on the original circle. Using Y as the center, and YP as the radius, draw an arc (not shown), which locates points B and E on the original circle. ABCDE is the desired regular pentagon.

To Prove: *The short-chord (YD) and the mid-diagonal (YE) are both in the required ratio to the radius (OY) so that the pentagon can be regular.* (See "The Ratios of a Pentagon" on the previous page.)

Proof: If we let OY = 1, then NK = OK = ½ ; YK = $\sqrt{5}/2$; YD = YN = $\frac{\sqrt{5}-1}{2}$

Therefore YD : OY = $\frac{\sqrt{5}-1}{2}$: 1 which is also **OY : YD =** $\mathbf{\frac{\sqrt{5}+1}{2}}$ **: 1 = Φ : 1**, as required.

YE = YP = ½ + ½ + $\frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2}$. Therefore **YE : OY =** $\mathbf{\frac{\sqrt{5}+1}{2}}$ **: 1 = Φ : 1**, as required. *Q.E.D.*