Appendix C – Euclid’s *Elements*.

**Selected Proofs** from *The Elements, Book I*

**Theorem 1** *Construction of an equilateral triangle, given one side.*

**Proof:**
1. Given line AB.
2. With A as center and using AB as the radius draw circle BCD (see drawing). With B as center and using AB as the radius draw circle ACE.
3. From point C, where the two circles intersect, draw lines to both points A and B.
4. Since point A is the center of circle BCD, AC \( \cong \) AB
   Since point B is the center of circle ACE, BC \( \cong \) AB
5. AC \( \cong \) BC
6. AC \( \cong \) AB \( \cong \) BC  \( \therefore \) \( \triangle ABC \) is equilateral.  Q.E.D.

**Theorem 5** *In an Isosceles triangle, (a) the base angles are equal to one another, and (b) if the two sides are extended, then the angles under the bases will be equal to one another.*

**Proof:**
1. Given \( \triangle ABC \) is an Isosceles triangle. Let AB and AC be the equal sides.
2. Extend sides AB and AC to D and E, respectively.
3. Choose point F at random on BD. Cut off AE at G, such that AG \( \cong \) AF.
4. Join the lines FC and GB.
5. Regarding \( \triangle AFC \) and \( \triangle AGB \):
   they share \( \angle GAF \); AF \( \cong \) AG (step 3); AB \( \cong \) AC (step 1)
   \( \therefore \) \( \triangle AFC \cong \triangle AGB \);
   FC \( \cong \) BG;  \( \angle ACF \cong \angle ABG \);
   and \( \angle AFC \cong \angle AGB \)
6. Since AF \( \cong \) AG and AB \( \cong \) AC then BF \( \cong \) CG
7. Regarding \( \triangle ABFC \) and \( \triangle CGB \):
   \( \angle BFC \cong \angle CGB \) (same as \( \angle AFC \cong \angle AGB \), step 5)
   BF \( \cong \) CG (step 7) and FC \( \cong \) BG (step 5)
   \( \therefore \) \( \triangle ABFC \cong \triangle CGB \), \( \angle BCF \cong \angle CBG \), and
   \( \angle CBF \cong \angle CGB \)  Q.E.D. (for part b)
8. Since \( \angle ACF \cong \angle ABG \) (step 5) and in these angles \( \angle BCF \cong \angle CBG \) (step 7)
   Then the remaining angles are equal :
   \( \therefore \) \( \angle ABC \cong \angle ACB \)  Q.E.D. (for part a)

---

1 All of the proofs listed here are the result of me re-wording T. L. Heath’s translation of *The Elements* (Dover Publications, 1956).
Appendix C – Euclid’s *Elements*.

**Selected Proofs from The Elements, Book I** (continued)

**Theorem 9**  *Construction of an angle bisector.*

*Proof:*
1. Given $\angle BAC$ (to be bisected) with point D randomly on AB.
2. Let AC be cut off at E, such that AD $\cong$ AE
3. Draw DE.
4. Draw equilateral $\triangle DEF$ on DE.
5. Draw AF.
6. DF $\cong$ EF
7. $\angle DAF \cong \angle EAF$
8. $\therefore \angle BAC$ has been bisected. Q.E.D.

**Theorem 10**  *Bisection of a line.*

*Proof:*
1. Given line AB to be bisected.
2. Draw equilateral $\triangle ABC$ on AB
3. Draw the bisector CD of $\angle ACB$
4. $\angle ACD \cong \angle BCD$
5. AC $\cong$ BC
6. $\triangle ACD \cong \triangle BCD$ and AD $\cong$ BD
7. $\therefore$ AB has been bisected. Q.E.D.

**Theorem 13**  *Supplementary Angle Theorem (Y Theorem).*  *If two adjacent angles form a straight line, then the sum of the angles is equal to two right angles.*

*Proof:*
1. Given line AB set up on line DC.
2. If $\angle CBA \cong \angle ABD$ then they are two right angles.
3. If these two angles are not equal, then draw BE from point B and perpendicular to line DC.
4. $\angle CBE$ and $\angle DBE$ are right angles.
5. $m\angle CBE = m\angle ABC + m\angle ABE$
6. $m\angle CBE + m\angle DBE = m\angle DBE + m\angle ABC + m\angle ABE$
7. $m\angle DBA = m\angle DBE + m\angle ABE$
8. $m\angle DBA + m\angle ABC = m\angle DBE + m\angle ABE + m\angle ABC$
9. $m\angle CBE + m\angle DBE = m\angle DBA + m\angle ABC$  
   (steps 6 & 8)
10. Because $\angle CBE$ and $\angle DBE$ are both right angles (step 4), $\angle DBA$ and $\angle ABC$ together form two right angles [they are supplementary]. Q.E.D.
Selected Proofs from *The Elements, Book I* (continued)

**Theorem 15** (X Theorem) *Vertical angles are equal.*

*Proof:*
1. Given lines AB and CD intersecting at E.
2. The sum of $\angle$CEA and $\angle$AED is equal to two right angles.
3. The sum of $\angle$CEA and $\angle$CEB is equal to two right angles.
4. The sum of $\angle$CEA and $\angle$AED is equal to the sum of $\angle$CEA and $\angle$CEB.
5. $\angle$AED $\cong$ $\angle$CEB (Similarly, it can be proven that $\angle$CEA $\cong$ $\angle$BED)  Q.E.D.

**Theorem 23** *Copying an angle.*

*Proof:*
1. Given angle DCE to be copied to point A on the line AB.
2. Let DE be drawn.
3. By using the three lines CD, DE, and CE construct the triangle AFG such that CD = AF, CE = AG, and DE = FG.
4. Since the three sides of the triangle AFG are equal to the three sides of the triangle CDE, then the angle DCE is equal to the angle FAG.  Q.E.D.

**Theorem 27** *If two lines are cut by a transversal, and alternate interior angles are equal, then the lines are parallel.*

*Proof:*
1. Given that EF falls on AE and CD, and $\angle$AEF $\cong$ $\angle$EFD.
2. Assume that AB and CD meet at point G, in the direction of B, D.
3. Then in $\triangle$EFG, the exterior angle ($\angle$AEF) is equal to an interior and opposite angle ($\angle$EFG), which is impossible.
4. $\therefore$ the assumption (step 2) is false.  AB and CD cannot meet in the direction of B, D.
5. Similarly, it can be shown that AB and CD cannot meet in the direction of A, C.
6. AB and CD do not meet in either direction, therefore they are parallel.  Q.E.D.
Appendix C – Euclid’s *Elements*.

**Selected Proofs from The Elements, Book I** (continued)

**Theorem 29** If two parallel lines are cut by a transversal, then the alternate interior angles are equal, the corresponding angles are equal, and the same-side interior angles add to two right angles.

[Note: This is the first theorem where Euclid uses the fifth postulate.]

Proof:
1. Given parallel lines AB and CD, with line EF falling on them.
2. Assume that $\angle AGH$ and $\angle GHD$ are not equal, and that $\angle AGH$ is larger. So, $\angle AGH > \angle GHD$.
3. $\angle AGH + \angle BGH > \angle GHD + \angle BGH$
4. $\angle AGH + \angle BGH = \text{two right angles}$
5. two right angles $> \angle GHD + \angle BGH$
6. Because the sum of $\angle GHD$ and $\angle BGH$ is less than two right angles, lines AB and CD must meet.
7. But lines AB and CD cannot meet.
8. Steps 6 and 7 are in contradiction, so our assumption (step 2) must be false, and $\therefore \angle AGH \cong \angle GHD$
9. $\angle AGH \cong \angle EGB$
10. $\therefore \angle EGB \cong \angle GHD$
11. $\angle EGB + \angle BGH = \angle GHD + \angle BGH$
12. $\angle EGB + \angle BGH = \text{two right angles}$
13. $\therefore \angle GHD + \angle BGH = \text{two right angles}$  Q.E.D.

**Theorem 32** In any triangle, (a) Any exterior angle is equal to the sum of the two opposite interior angles, and (b) The three interior angles add to two right angles.

Proof:
1. Given $\triangle ABC$
2. Extend BC to D
3. Draw CE parallel to AB
4. Since AB is parallel to CE, and AC transverses both of them, $\angle ACE \cong \angle BAC$
   and $\angle ECD \cong \angle ABC$
5. $\angle ACE + \angle ECD = \angle BAC + \angle ABC$
6. $\angle ACE + \angle ECD = \angle ACD$
7. $\angle ACD = \angle BAC + \angle ABC$  Q.E.D. (for part a)
8. Adding $\angle ACB$ to both sides of equation: $\angle ACD + \angle ACB = \angle BAC + \angle ABC + \angle ACB$
9. But $\angle ACD$ and $\angle ACB$ are adjacent angles and form the straight line BD, therefore they are equal to two right angles.
10. $\therefore \angle BAC + \angle ABC + \angle ACB$ is also equal to two right angles.  Q.E.D. (for part b)
Selected Proofs from *The Elements, Book I* (continued)

**Euclid's Proof of the Pythagorean Theorem**
(Theorem I-47)

1. Given: \( \triangle ABC \) is a right triangle, with \( \angle BAC \) a right angle.
2. Construct a square on each of the 3 sides of \( \triangle ABC \).
3. Draw \( AL \) parallel to \( BD \).
4. Draw lines \( AD \) and \( FC \).
5. (a) \( \angle DBC \) & \( \angle FBA \) are both right angles.
   (b) \( \angle DBC \equiv \angle FBA \)
   (c) \( \angle DBC + \angle ABC = \angle FBA + \angle ABC \)
   (d) \( \angle ABD \equiv \angle FBC \)
6. (a) \( BD \equiv BC \) and \( AB \equiv FB \).
   (b) \( \triangle ABD \equiv \triangle FBC \) because \( BD \equiv BC \) and \( AB \equiv FB \) and \( \angle ABD \equiv \angle FBC \) (step 6).
7. (a) \( \angle BAG \) is a right angle.
   (b) \( \angle BAC \) and \( \angle BAG \) are adjacent and both right angles, so \( CA \) is in a straight line with \( AG \).
   (c) \( \angle BAC \equiv \angle FBA \)
   (d) \( CG \) is parallel to \( FB \).
   (e) [The area of] square \( GB \) is twice [the area of] \( \triangle FBC \), because they have the same base \( FB \) and lie between the same parallels \( FB \) and \( GC \).
8. [The area of] parallelogram \( BL \) is twice [the area of] \( \triangle ABD \), because they have the same base \( BD \) and they lie between the same parallels \( BD \) and \( AL \).
9. \( \triangle FBC \equiv \triangle ABD \), therefore twice [the area of] \( \triangle FBC \) is equal to twice [the area of] \( \triangle ABD \).
10. [The area of] square \( GB \) is equal to [the area of] parallelogram \( BL \).
11. Similarly, if lines \( AE \) and \( BK \) are drawn, parallelogram \( CL \) can be proven equal to square \( HC \).
12. The sum of [the areas of] squares \( HC \) and \( GB \) is equal to the sum of [the areas of] parallelograms \( CL \) and \( BL \).
13. [The area of] the square \( BE \) is equal to the sum of [the areas of] parallelograms \( CL \) and \( BL \).
14. \( \therefore \) [The area of] the square \( BE \) is equal to the sum of [the areas of] the squares \( GB \) and \( HC \).

*Q.E.D.*